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THE MATHEMATICS TEACHER

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MATHEMATICS AND MODERN LIFE[1]

BY GEORGE B. OLDS

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It is always a pleasure to face and to greet a group of teachers, and especially teachers of mathematics. I find myself peculiarly fortunate this evening in the privilege of addressing a national group representing that portion of the teaching profession which, according to Commissioner Tigert, has to do with the training of some ten million boys and girls. Education has many definitions. I recall very vividly Huxley's. He characterized the aim of education as, "not the survival of the fittest but the making of as many as possible fit to survive." In one of the dark recesses of my memory I half recall another definition. "Education represents a systematic attempt by the adults of one generation to make their children wiser and better than themselves." In other words, education is the price of progress —and as teachers are incurably optimistic, we who are assembled here tonight believe in progress, are thoroughly persuaded that tomorrow will brighter than today.

Of this process which Huxley so epigrammatically characterized, this process by which man is ever pursuing a receding and yet higher goal—a sort of asymptotic approach to the best; we, you and I, are or have been the instruments. I can think of no nobler calling. A week or so ago I was addressing a group of ministers and conceded, at the outset, that apart from certain static conditions as to theology and creed their profession stood high. I reminded them, however, that to the profession of a teacher belongs the primacy. The history of the world revolves about a man who gloried in being called *magister* by his *discipuli*. The *Master* "looked upon the young man and loved

[1] Address at the Boston Meeting of the National Council of Teachers of Mathematics, February 24, 1928.

him." I speak of teaching as a profession, which some one has defined as a "trade shot through with the spirit of service."

Yes, I feel very much honored to be here tonight. If Sir Isaac Newton's Third Law prevails in this room—namely, that "action and reaction are equal, contrary, and simultaneous," the choice of your Committee will be justified, for it is an inspiration to feel the contact with colleagues whose trade is that of makers of men.

There have been times, when mathematics was fashionable, when it was quite the mode to refer to it in social circles. The story is told that, shortly after the invention of the differential and integral calculus, the ladies and gentlemen of the French court were hardly at their best unless they could speak casually of "infinity and infinitesimals." In fact, the street gamins of Paris sang a doggerel of which the refrain was "Les infiniments petits." Another instance of such casual social interest in the profundities of the science came to the forefront when mathematicians began to talk of the non-Euclidian geometry and the fourth dimension. There was something fascinating about the thought that parallel lines might, after all, meet, or that there might be an additional dimension in a space, that there might be curvature, hence finiteness of space.

In one of the early numbers of the *Quarterly Journal of Mathematics*, published by The Johns Hopkins University, Simon Newcomb, one of perhaps the most distinguished mathematical astronomers of his time, contributed an article in which he proved that the existence of a fourth dimension in space would make it possible to separate two links of an anchor chain without penetrating the steel, or to turn an egg wrong side out without breaking the shell. It was inevitable that such extraordinary performances should be of absorbing social interest. In some recent times we have had a revival of this social interest in our science. I question whether there is a member of our community, man or woman, or even a boy or girl, that does not respond in a more or less lively way to a suggestion of the Einstein theory—a theory whose basis is more abstruse and difficult of apprehension than even the n dimensions of space. Odd biographical facts are also of interest to the curious, *e.g.*, Newton and his dog, Cardan and his suicide.

The so-called mathematical puzzles have entertained men

through the centuries. Books containing puzzles of this kind have appeared from age to age, and are among the so-called amenities and curiosities of mathematics. The insoluble problems which men have always thought could, after all, be solved, namely, the trisection of an angle; the squaring of the circle and the duplication of the cube have also been of great popular interest. Such uncanny things as zero and infinity have likewise a mystical interest for the laity. And the paradoxes which could arise from their use have been a focus of almost passionate controversy.

On the lighter side then, mathematics has ever made its appeal to men and women—and makes that appeal today. May we pass now to the serious side of the subject suggested by your program.

Your president, or some one in authority, has taken the liberty of styling my talk of the evening "The Keynote Address." The choice was not mine, but I accept it. It is like a minister's text—it may serve as a point of departure. I am uncertain whether those responsible for the choice are musicians or politicians. For a moment, let us assume the former. George D. Olds is to sound the piano note A for the members of the orchestra. The distinguished speakers who follow will, tomorrow, discourse sweet music ending very properly with "Mathematics and Sunshine." It will be a case where (to paraphrase "Richard the Third") "the winter of your discontent will be made glorious summer by the sun of York." It seems simple, this giving of a keynote—but suppose the piano is out of tune or (more frightful still) suppose the player strikes a false note. Appalling! But, after all, not so appalling as in the days before cubist music came to be in vogue and modernists luxuriated in discords. The Keynote Address. Let us consider the other alternative, the political one. The thought is peculiarly appropriate this year, one of the years in which the initial impulse toward marriage undergoes a sex reversal, in which women have their way and in which we choose presidents, and then perhaps, too, women will have their way. Our eyes are already beginning to turn toward Kansas City and Houston (remember, pronounced "Hewston"), where in June spell-binding orators will warm the hearts of thousands of politicians up to the temperature of their bodies at the time.

One distinguished orator will be chosen as "Temporary Chairman" of the Convention and will make the "Keynote Address." In general, it cannot be precisely predicted what he will say—but of one thing we can be sure. Like "All Gaul," his discourse will be divided into three parts: (1) "Exordium," (2) "View with Alarm," (3) "Point with Pride." At Kansas City, the orator will view with alarm the insidious activities of the opposition jeopardizing prosperity and undermining our fundamental institutions and he will point with pride to the benign activities and the high-minded record of those who, for nearly eight years, have carried the responsibilities of governance. At Houston—*mutatis mutandis*—we may listen to corresponding oratory.

It is in the above sense, again, that the talk you are so good as to listen to this evening is a Keynote Address. For the speaker has already completed an exordium and the remainder of what he has to say may roughly fall under two heads, "View with Alarm," "Point with Pride." Or, rather, he will do as the political keynoter usually does—be very brief in his alarm and put his soul into his pride. From my correspondence with your Committee, I assume, and even without anything explicit from its members, I *should* assume that the general theme of these meetings means an organized protest against a movement among a few of those who control education in our country to belittle the importance of our science even to the point of reducing the attention given to it in our secondary schools to a vanishing minimum. Some years ago, in this city of Boston, I attended a meeting of the Teachers of Mathematics in New England, at which the principal speaker was the man who, at the time, was Commissioner of Education for the Commonwealth of Massachusetts. The burden of his address was an attack on algebra as an integral part of a high school curriculum. In view of the ever-increasing pressure of the newer subjects of knowledge—especially of the natural and social sciences, something, he felt, must give way and his scapegoat was algebra. It was my impression at the time that the attack was only an entering wedge, and that, in the not distant future, geometry would be under fire too.

It is fairly sure—and this makes the alarm a more genuine one—that the disappearance of mathematics from the secondary

schools, especially of algebra, would find a cordial welcome for most of our pupils. Mathematics is abstract—life is concrete; the boys and girls feel this, though their progress in mature thinking may not have attained the point where they can adequately express it. Mathematics is hard—mathematics is uninteresting—yes, they are sure of that and have no difficulty here in expressing their feelings. Have they not, from the kindergarten up, been softly cushioned against the disagreeable bumps of training? These boys and girls feel, too, that the mathematical courses of the four years have little or no contact with the life they are to lead after leaving school. The distinguished speaker in question urged, as one of his strongest points, that few men have the faintest remembrance of the algebra they learned in school. Someone, by the way, had the audacity to ask him afterwards how many men remembered the chemistry they have studied, the science he would have substituted for the mathematical requirement.

I mention this address as only a single typical instance, but you will recall many talks and articles like it. The classics went, or lost ground, many years ago. It is time for our science to be on its guard lest it undergo the same fate. Yes, one whose life work has been dedicated to the teaching of mathematics may well "view with alarm" certain insidious tendencies, but that alarm should pass like a morning mist when one turns to the past, thinks of what mathematics has meant to the world of men and women, thinks of the splendid *vis viva*, the irresistible momentum of its history. Then, we shall have "mathematics with sunshine" indeed.

It is the patriarch of sciences. There is good evidence that number symbols antedated in human history word signs. The Rhind papyrus, discovered in Egypt some twenty years ago, furnishes the proof that possibly three thousand, certainly two thousand years, B.C. the Egyptians were aware of certain fundamental operations with numbers, with the solution of a simple equation involving one unknown quantity, and with many geometrical facts. Among others, Ahmes, its author, gives an excellent approximation to the value of pi, much better than that of the Hebrews who, you may recall from your Sunday School lessons, recorded that a certain cylindrical vessel in Solomon's Temple was to be one cubit in diameter and three cubits in

circumference. For the purpose of mensuration of areas and of solids it was inevitable that geometry should come into being.

You know how far the Greeks carried the science. They fell down badly in their arithmetic and of course, then, in algebra, being handicapped by their alphabetic notation, but about 300 B.C. they produced a textbook of geometry, that with minor modifications (few of them improvements), has remained the textbook of the world to this day. In my secondary school days geometry and Euclid were synonymous terms. The Romans added little and there was a period of Medieval darkness and stagnation. Then came the renewed interest born of the Renaissance, the transformation of rhetorical algebra into symbolic, the great golden age of mathematics culminating in and proceeding from Newton, the epoch-making yeasting from the leaven of the imaginary unit with the magnificent result which has characterized the study of the theory of functions for the past hundred years. The layman thinks mathematics has been static, unchangeable, yet Miller's investigation has shown that the printed contributions to mathematical theory since 1850 bulk larger than the total of all that had gone before.

That reminds me of the famous saying of Leibnitz, Newton's great rival in his claim to the invention of the differential and integral calculus. Leibnitz was on his deathbed—his father confessor was at his side and asked him whether he wished to put on record any last words. "Yes," said Leibnitz, "it's about Newton. Taking mathematics from the beginning of the world through Newton's time, what he has done is much the greater half."

In the course of this long and majestic history mathematics has laid the foundation for an exact astronomy, made engineering possible, became the life blood of physics and thereby, of course, of chemistry, penetrated biology with a differential equation (I saw it myself, in an article on genetics), helped President Hadley in his theory of railway tariffs, supplied an auxiliary framework for economics, rationalized life insurance, and has become the statistician's inseparable ally.

With all this record of touch with life, how can the critics say "Cui bono," "What's the use!" Indeed there is really no useless knowledge. Facts collected and systematized, principles discovered and welded into unity have eternal value. When

Albert Michelson was asked why he had all through life been taking such infinite pains in experimenting with light, why his revolving mirrors, why his interferometer, his unremitting toil on microscopic minutiæ of tests and measurements, why all this attention to details so far from the worldly point of view—his reply was: "Why do I do all this? Why, for the fun of it!" That is the keynote of scientific progress. For the fun of it! Knowledge needs no justification; wisdom is justified of her children.

Moreover, if it were not for just this spirit where would material civilization be today. Michael Faraday discovered a marvelous relation between moving electric circuits and magnets. "What's the use of it?" said a high-born dame when she witnessed his experiments. "What's the use of a new-born baby?" said Faraday. "What's the use?" said Gladstone. "In a few years, you will be taxing it," was his reply. The outcome was the dynamo.

Some years later there was working in Helmholtz's physical laboratory a young man by the name of Hertz. Clerk Maxwell had put a mathematical spirit into the body of Faraday's discoveries, with the logical conclusion that electricity must be assimilated to light as a wave phenomenon of the ether. As Maxwell had furnished mathematical life spirit for Faraday's body of science, so Hertz now clothed the mathematical soul that Maxwell had brought into being with a new body, shall we say with St. Paul, "a spiritual body." He discovered the very waves that Maxwell had mathematically predicted. A thrill went through the world of science; but it seemed a far cry to utility, yet within a dozen years Marconi had used Hertz's discovery to give to the world wireless telegraphy, and every room in this Statler Hotel with its radio apparatus is eloquent of the outcome. No, there is no such thing as useless knowledge, either on the plane of so-called real life by which men mean livelihood or on the higher plane of "*real,* real life."

But there is something more vital in education than knowledge. Making men fit to survive, making our children wiser and better than ourselves means vastly more than imparting knowledge, though that is the oxygen, the sustenance of the mind. The higher aim is training. Some one has said that "education is what is left over after we have forgotten what we

learned." The man's mathematics and his chemistry may be in the limbo of departed knowledge. It is the discipline of good teaching and thinking that has remained.

In recent years there has been a movement inaugurated by certain professional educators to deny that there is any such thing as the transfer of discipline; in other words, they refuse to believe that the training acquired by the study of any one subject can serve a pupil in the pursuit of any other, and, above all, that it can be of service to a man in meeting the emergencies of actual life. To deny that the mind can be thus trained is like saying that the tackling dummy is of no service to the football player, or that the training of a child in well devised bodily exercises has no bearing on his subsequent life. In fact, if the theory advanced that there is no transfer of training be carried to its logical conclusion, it would lead to a *reductio ad absurdum;* in other words, there would be no justification for education at all, for schools are fundamentally the places of training for life by means of a discipline that except for the specialist will rarely in its precise form be met later.

It is in this apparatus for training that mathematics is supreme; there is no other branch which can be listed in the things taught in our schools and colleges that can compare with it in its capacity to train the mind. We think first of all of the reasoning process; mathematics is nothing if not logical. We make certain postulates or assumptions—axioms we call them—which we agree to accept, and from these by the process of pure logic deduce certain conclusions which have the character of universal truth. Some one has said that Newton's *Principia* could be read with interest and profit by the mathematicians of Mars, if it could be rendered into their vernacular. All other forms of science and thinking lead merely to that which is more or less probably true. Mathematics is the only science which can develop a knowledge of absolute truth.

Another form of the training given by mathematics not often thought of is that of clarity and directness of expression. It might seem to the layman a long way from mathematics to the use of one's vernacular, and yet there is a most marked interdependence. A number of years ago I had a pupil at Amherst now at the head of the English Department of one of our best eastern colleges, and one of the most brilliant teachers of English

in the country. He has, also, a style which is an inspiration to his pupils, a daily example of what is best in oral and written expression. At college he took the first required year in mathematics and elected calculus in the second year. At that time he had made up his mind to devote his life to the study and teaching of English. I assumed, of course, that he would take no more mathematics and was surprised to learn from a colleague that he was determined to continue the subject until the end of his college course. When I asked him his reason he said that he wished the kind of training that mathematics gave in order to do two things; one, to prune, and the other, to clarify his English style. He felt that he was too diffuse, too exuberant in his way of expressing himself and too obscure, at times. Through the close reasoning and clear expression demanded by courses in mathematics he hoped to effect a cure. That he was satisfied in the end with the choice he had made became evident, when, some years later, he wrote an article recounting his experiences in this study of an abstract science, and recommending such a course to those who suffered as he thought he suffered in the use of the English language.

I have also in mind an illustration of the way in which the training in the use of the reason may serve a man in after life. I refer to a very brilliant example of a man whose name is today known all over the United States; the one who was really responsible for Lindbergh's many flights, serving as they are doing to cement and make cordial the international sentiments of America. The man in question took four years of mathematics and then went into the study of law. Later he became the legal member of an important financial firm. I remember calling on him soon after he assumed this new responsibility, and asked him whether he had any occasion to use his mathematics. "Arithmetic and the technical formulæ of the various branches in mathematics, very rarely," he said, "but its logic, method, and way of reasoning, almost every day. A problem connected with the experience of the firm I find myself facing as a mathematician faces his problem. Certain things are given, a logical method supervenes, a perfectly definite form of deductive reasoning and an inevitable conclusion must follow." Yes, mathematics as a tool for training can yield to no other branch of knowledge handled in the curriculum of an educational institution.

But one thing more! Beyond the imparting of knowledge, the equipment of the mind—beyond training, the giving of a cutting edge to the mind—there is something far more important among the aims of education, namely, the implanting of ideals. The whole history of the human race proves that ideals are the only real things; for it is they that make up character. Our science is eminently adapted even to this, for it aims at truth and beauty. Keats, in his *Ode to a Grecian Urn*, makes truth and beauty identical. As has already been said, the essence of mathematics is truth and in the apprehension and expression of truth it exhibits marvelous beauties. The very perfection of its methods and results meets an æsthetic demand. There are many things in the science—even after scores of years have made me a seasoned veteran—that today send a thrill through me when I contemplate them.

As an example, the fundamental fact that by considering the conic sections as the projection of a circle their properties may be determined. How often have I repeated the boyish experiment of projecting the shadow of a dinner plate on a white wall, watching the ellipse elongate and elongate until, as the ray of light grazed the outer edge of the plate and became parallel to the wall, the ellipse suddenly passed into a parabola, then at the next move into one branch of a hyperbola.

Then Klein's magnificent unifying principle by which he would make the propositions of geometry only the invariants of a group of substitutions. That word "invariant," how wonderful, hence how beautiful, for the sublime is neighbor to the beautiful, Burke tells us.

You know that in analytic geometry we deal with a general equation of the second degree in two variables as the algebraic representation of a conic section. In general, the curve has a highly unsymmetrical position in the plane. If an ellipse, say, its center is out in the wilderness, perhaps in the third quadrant, and its axes are at seemingly hopeless angles with the fixed axes of coordinates. We transform the curve so as to obtain central symmetry; we transform again so as to bring the axes into coincidence with the main thoroughfares. The resulting equation is a thing of beauty, but looks utterly unlike its original. Not a single coefficient remains to tell the story of its birth. Yet there are two functions of these coefficients that remain as

unchanged as solid rock, two determinants of the transformation. Taken together they determine an intrinsic property of the ellipse, namely, its area. We call them the invariants of the transformation. These invariants remind me of a familiar experiment in physics. You know it well. Place a sheet of cardboard over a horseshoe magnet; sprinkle iron filings on the cardboard and there will appear at once curves from pole to pole of the magnet. Shake the cardboard and the filings will go into a fresh chaos. But the curves remain. We call them lines of force, but they are really the invariants of a physical transformation.

Pi and the exponential e and a logarithm are sorry spectacles when put into the straight jacket of decimal fractions, but they become luminous when written

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots,$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right),$$

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

The formula which is the basis of DeMoivre's theorem,

$$e^{i\phi} = \cos\phi + i\sin\phi$$

binds exponentials and trigonometric functions into a perfect whole. Let us put $\phi = 2\pi$ in the formula and we get

$$e^{2\pi i} = 1,$$

an irrational base wearing as its exponential visor a product of an irrational by the imaginary unit, the whole mystical combination turning out unity! !

$$\left(1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots\right)^{8(1-\frac{1}{3}+\frac{1}{5}-\cdots)\sqrt{-1}} = 1 \,!\,!\,!$$

The problem of three bodies in mathematical astronomy is summed up by Kirchhoff in his great work on mechanics (as early as the twelfth page) in ten equations all of the simplest type and all derived from Newton's simple law

$$F = \frac{m}{r^2}.$$

Is not this the very quintessence of scientific beauty!

One of the most striking facts that emerge as one reads mathematical history or the biographies of mathematicians is the intimate relations of mathematics and art. The basis of Greek architecture was geometric. Delicate curves were introduced into the lines of columns and porticoes (architects called the process Éntasis) cunningly devised to create in the eye of the observer the effect of straightness, yet not a single line is straight. In the school of Pythagoras music and mathematics are interwoven. In modern music, both the diatonic and the chromatic scale form a mathematical series. The dimensions of strings and organ pipes determine the pitch of music by entering into a differential equation of the simple type,

$$\frac{d^2x}{dt^2} = -ax.$$

It has been said that some of Bach's fugues could have been mathematically constructed.

In writing of Maxwell and Kirchhoff, Bolzmann says of the latter: "Kirchhoff's work has extraordinary beauty. The graces do not flee when integrals appear. Merely by his simplicity, the indispensableness of every word, every letter, every subscript, the mathematician comes as an artist close to his Creator. His closest companion is the composed of symphonic music '*ultima se tangunt.*'" Ultimates touch!

Of Maxwell he says: "The English mathematicians are characterized by the power of dramatic presentation. This is especially true of Maxwell. At first (speaking of his Kinetic Theory of Gases) the variables which represent velocities appear upon the stage. On one side the equations of the initial condition of the gas take their place; on the other side of the stage we see the equations of the central motions. The evil demon V appears. Ever higher and higher, like great waves, heaves the chaos of formulæ. Of a sudden the words ring out "Put $n = 5$." The demon V vanishes; just as in music a wild tumult of the double basses confounding and dominating everything may suddenly become silent and as if with a magician's touch order reigns in chaos. Obediently the formulæ yield result on result, when suddenly we face the astounding finale, the conditions of equilibrium of a heavy gas—and the curtain falls."

It seems, too, a far cry from mathematics to poetry. When I

was at the University of Cambridge, a few years ago, one of the dons of Christ College, proud of his Alma Mater, took me to the great hall which every English college lovingly exhibits. He instantly fixed my attention on two portraits, one of Darwin, the other of Milton, the scientist and the poet. Cambridge is the great nursery of mathematicians, as you know, but it is not so well known that it is the academic birthplace of poets. The University that gave to the world Newton and Maxwell and Kelvin also mothered Spencer, Marlowe, George Herbert, Milton, Cowley, Gray, Dryden, Coleridge, Wordsworth, Byron, Tennyson and countless minor poets.

Some years ago Amherst College was host to Neils Bohr, of Copenhagen, at the time the most distinguished mathematical physicist in the world. He had said the latest and, some enthusiasts affirm, the last word on the structure of the atom. He is responsible for the solar system theory of this structure; quiescence when the constituent electrons remain at home, each in its own orbit about the proton, but radiation when these electrons jump from orbit to orbit like undisciplined boys. Few of us were able to understand his theory, but everyone caught the contagion of his presence. Like all communities, a college community finds the sudden appearance of a man of genius a disturbing influence. Bohr had not been with us many days before he expressed a strong desire to meet Robert Frost, a highly prized possession of the college. One Sunday evening I succeeded in getting the two men together at my fireside. There were a few awkward moments, some oppressive silence, then the talk began. It was ended only by my wife's opportune appearance to invite the two men into the dining room. They had been in touch with each other in a trice, the man of scientific imagination and the man of poetical imagination. As Bolzmann said, *"ultima se tangunt"*—mathematics and poetry!

But, alas, I seem to have wandered far afield from the general thought of these meetings—Mathematics and Modern Life. Yet my wandering has not been without a purpose. Modern life is, after all, the world's life. It may change its emphasis and gain in complexity. It may apparently become more utilitarian; life may seem to the pessimist to be translated into livelihood; we may seem to forget at times that while we all aim to keep the wolf from the door, it would not be worth the effort to banish him if there were not something higher and better than

mere existence. The fundamentals of true living are eternal verities. The relation of mathematics to modern life is its relation to the life of all times. What we teachers need is to become and to remain aware of the sweep, the grandeur, the universality of the science we love—and to venerate it. Let us read its history, the biographies of its great men of genius, biographies such as Appell's *Life of Poincaré* or Brewster's *Life of Newton*. Let us get close to that tragic night in the life of Galois, the young Frenchman only twenty years of age, who in the dark hours just before the morning duel in which he felt certain of losing his life, wrote on scraps of paper a theory which, when put in order, was found to have revolutionized modern algebra.

Much of what has been referred to in illustrating the power and beauty of mathematics cannot, of course, be made intelligible to our pupils; but, all the same, they will feel the atmosphere of our own interest, catch the contagion of our contact with the highest and the best. Just as in the case of Bohr we could not follow him as he moved amid the intricacies of his theory, but his enthusiasm and evident genius seized us all with a grip that we shall never forget. For the sake of our pupils we must live from time to time on Parnassus and dwell with our muse.

In conclusion, let me quote from a book I studied more than fifty years ago, Bartlett's *Mechanics*. The words are part of the Preface. The majestic significance of our science in catholicity and impressing character can be summed up no better.

"It embraces alike, in their reciprocal action, the gigantic and distant orbs of the celestial regions, and the proximate atoms of the ethereal atmosphere which pervades all space and establishes an unbroken continuity upon which its Divine Architect and Author may impress the power of His will at a single point and be felt everywhere. Astronomy, terrestrial physics, and chemistry are but its specialties; it classifies all of human knowledge that relates to inert matter into groups of phenomena, of which the rationale is in a common principle; and in the hands of those gifted with the priceless boon of a copious mathematics, it is a key to external nature."

So, my friends, if at times a panic seems to seize us, if now and then we view with alarm the fate of our cherished science, let us look at it upon the lofty pedestal where it belongs—pointing with pride to a science which cannot be dethroned.

FINDING PLANE AREAS BY ALGEBRA

By JOS. B. REYNOLDS,

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The wide-awake teacher of mathematics is ever on the alert for opportunities of vitalizing his subject. He has a keen eye on the lookout for applications which will bring the day's work for the pupil into relation with the field of experience, or which will make use of knowledge acquired in one field to further efforts in others. To some students applications of a subject or method outside of those found in the text studied is a never-failing stimulus to renewed thought and effort. The beginning of a life of research is frequently to be found in such small revelations.

It is difficult to find applications of certain parts of algebra in elementary fields but too little use has been made of it in geometry. It seems reasonable that the secondary school student should be taught as much as possible to make a tool of his algebra regardless of whether his education is to end with the high school or not. To teach the subjects of algebra and geometry in such a way that students conceive of them as two distinct methods of reasoning seems a poor preparation for applying either to the problems of life or to the types of analysis used in more advanced courses.

The summation of series by mathematical induction or by the method of undetermined coefficients is apt to seem very much detached from reality to the beginner. However, the limit of a summation is the foundation principle of the integral calculus which has so many useful applications. It is the aim of this article to give some applications of algebra by showing how to find a number of plane areas by means of the limiting values of the summations of series. We shall show how to find certain areas by means of known sums of series and, conversely, how to sum certain series by means of known areas.

From algebra we have the following summations:

$$s_1 = 1 + 2 + 3 + \cdots + n = n(n+1)/2,$$
$$s_2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6,$$
$$s_3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = n^2(n+1)^2/4.$$

Designating by S_1 the limiting value of s_1/n^2 as n increases beyond bound (as $n \to \infty$) we have

$$S_1 = \text{Lim}_{n\to\infty} n(n+1)/2n^2 = \text{Lim}_{n\to\infty}(1 + 1/n)/2 = 1/2.$$

In a similar manner, designating by S_2 the limit of S_2/n^3 as n increases beyond bound, we have $S_2 = \text{Lim}_{n\to\infty} n(n+1)(2n+1)/6n^3 = \text{Lim}_{n\to\infty}(1 + 1/n)(2 + 1/n)/6 = 1/3$. In like manner $S_3 = 1/4$. Upon inspecting the values of S_1, S_2, and S_3 the student will be led to expect that $S_4 = 1/5$ and, in general, that $S_m = 1/(m+1)$.

The last relation may be proved as follows:

Assume

$$(1)\quad s_m = 1^m + 2^m + 3^m + \cdots + n^m = An^{m+1} + Bn^m + Cn^{m-1} + \cdots + N;$$

in which $A, B, C, \cdots N$ are undetermined numerical coefficients and m is any positive integer. Replacing n by $n+1$ in (1), that is including one more term in the series, we have

$$(2)\quad 1^m + 2^m + 3^m + \cdots + n^m + (n+1)^m = A(n+1)^{m+1} + B(n+1)^m + C(n+1)^{m-1} + \cdots + N.$$

Subtracting (1) from (2) we find

$$(n+1)^m = A[(n+1)^{m+1} - n^{m+1}] + B[(n+1)^m - n^m] + C[(n+1)^{m-1} - n^{m-1}] \cdots,$$

and upon equating coefficients of n^m, after expanding by the Binomial Theorem or taking out the factor $(n+1) - n$, we have $A = 1/(m+1)$. Now by (1)

$$s_m/n^{m+1} = A + B/n + C/n^2 + \cdots + N/n^{m+1},$$

and therefore

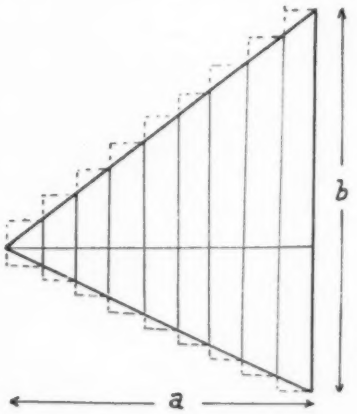
$$S_m = \text{Lim}_{n\to\infty} s_m/n^{m+1} = A = 1/(m+1).$$

We shall now proceed to apply these results to finding plane areas beginning with the simple case of the area of a triangle.

1. To find the area of a plane triangle of base b and altitude a.

Let segments of lines parallel to the base as shown in the figure divide the altitude into n equal parts. Let y be the length of a segment at a distance x from the vertex. We shall then have $y/x = b/a$. Construct rectangles on each of the segments y_1, y_2, y_3, $\cdots$, y_n as bases. Each rectangle will have an altitude a/n

and the sum, A^1, of their areas is $(a/n)[y_1 + y_2 + y_3 + \cdots + y_n]$. But $y_1 = bx_1/a = b/n$, since $x_1 = a/n$; $y_2 = bx_2/a = (b/a)(2a/n)$



$= 2b/n$; $y_3 = 3b/n$; $\cdots$; $y_n = nb/n$; whence

$$A^1 = (a/n)[y_1 + y_2 + y_3 + \cdots + y_n] = (ab/n^2)[1 + 2 + 3 + n].$$

Now the limit, A, of the sum of the areas of the rectangles as n increases beyond bound is the area of the triangle; that is $A = \text{Lim}_{n\to\infty} A^1 = abS_1 = ab/2$.

2. To find the area of a trapezoid of bases b and $b^1 (b < b^1)$ and altitude a.

As before let the altitude a be divided into n equal parts by line segments of lengths $y_1, y_2, y_3, \cdots, y_n$. Then $y_1 = b + (b^1 - b)/n$; $y_2 = b + 2(b^1 - b)/n$; $y_3 = b + 3(b^1 - b)/n$; $\cdots$; $y_n = b + n(b^1 - b)/n$ and the sum A^1 of the areas of the rectangles with the y's as bases each of altitude a/n is

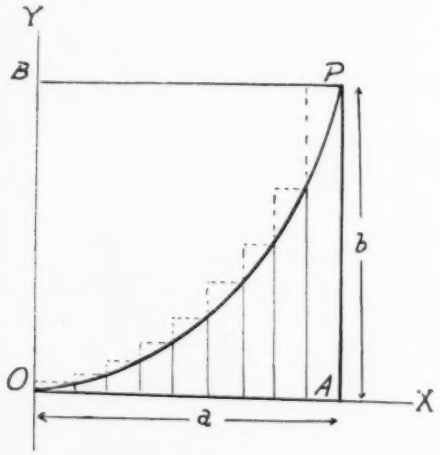
$$\begin{aligned} A^1 &= (a/n)[y_1 + y_2 + y_3 + \cdots + y_n] \\ &= (a/n)[nb + (1/n)(b^1 - b)(1 + 2 + 3 + \cdots + n)] \\ &= ab + a(b - b^1)(1 + 2 + 3 + \cdots + n)/n^2, \end{aligned}$$

and the area, A, of the trapezoid is given by $A = \text{Lim}_{n\to\infty} A^1 = ab + a(b^1 - b)S_1 = a(b + b^1)/2$.

3. To find the area of the parabolic spandrel of base b and altitude a (OAP in the figure).

The equation of the parabola in the position shown is of the form $x^2 = py$. Since when $x = a$, y must have the value b we

must have $a^2 = pb$, or $p = a^2/b$ which gives for the equation of the curve $y = bx^2/a^2$.



Draw line segments parallel to the base b dividing the altitude a into n equal parts. Let the lengths of these segments be y_1, y_2, y_3, $\cdots$, y_n, each being drawn from the x-axis, OA, to the curve. Then $y_1 = bx_1^2/a^2 = b/n^2$, since $x_1 = a/n$; $y_2 = bx_2^2/a^2 = b(2a/n)^2/a^2 = 4b/n^2$; $y_3 = 9b/n^2$; $\cdots$; $y_n = n^2b/n^2$. Construct rectangles upon these line segments as bases each of altitude a/n. Then the sum, A^1, of the areas of these rectangles is

$$A^1 = (a/n)[y_1 + y_2 + y_3 + \cdots + y_n]$$
$$= (ab/n^3)[1^2 + 2^2 + 3^2 + \cdots + n^2],$$

and the area, A, of the spandrel is

$$A = \text{Lim}_{n \to \infty} A^1 = ab\ S_2 = ab/3.$$

4. To find the area of a spandrel of the cubic parabola of base b and altitude a.

The required equation of the cubic parabola is $y = bx^3/a^3$. Proceeding in exactly the same manner as in the last case one easily finds the area, A, to be $abS_3 = ab/4$.

It now becomes clear that the area bounded by the curve $y = bx^m/a^m$, the x-axis and the line $x = a$, is

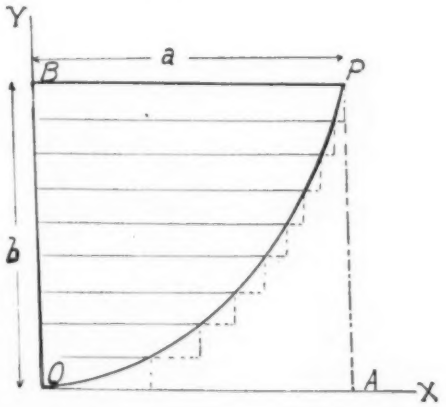
$$A = abS_m = ab/(m + 1).$$

We turn at this point to the converse problem of finding the limit of the sum of certain series from known areas.

5. To evaluate

$$S_{1/2} = \text{Lim}_{n\to\infty}[(1)^{1/2} + (2)^{1/2} + (3)^{1/2} + \cdots (n)^{1/2}]/n^{3/2}.$$

Consider the area of the parabolic segment OPB of base b and



altitude a. Let segments of lines parallel to the x-axis as shown divide the base b into n equal parts. Since the equation of the parabola is $y = bx^2/a^2$ the lengths $x_1, x_2, x_3, \cdots, x_n$ of these segments are $x_1 = a(y_1/b)^{1/2} = a(1/n)^{1/2}$, since $y_1 = b/n$; $x_2 = a(y_2/b)^{1/2} = a(2/n)^{1/2}$; $x_3 = a(3/n)^{1/2}$; $\cdots$; $x_n = a(n/n)^{1/2}$. Let rectangles be constructed upon these segments as bases, each of altitude b/n. Then the sum, A^1, of the areas of these rectangles is

$$\begin{aligned} A^1 &= (ab/n)[x_1 + x_2 + x_3 + \cdots + x_n] \\ &= ab[(1)^{1/2} + (2)^{1/2} + (3)^{1/2} + \cdots + (n)^{1/2}]/n^{3/2}. \end{aligned}$$

The area, A, of the segment OPB of the parabola is equal to the area of the rectangle $OAPB$ less the area of the spandrel, OPA; that is $A = ab - ab/3 = (2/3)ab$. Also,

$$\begin{aligned} A &= (2/3)ab \\ &= \text{Lim}_{n\to\infty} ab[(1)^{1/2} + (2)^{1/2} + (3)^{1/2} + \cdots + (n)^{1/2}]/n^{3/2}, \end{aligned}$$

whence

$$S_{1/2} = \text{Lim}_{n\to\infty}[(1)^{1/2} + (2)^{1/2} + (3)^{1/2} + \cdots + (n)^{1/2}]/n^{3/2} = 2/3.$$

Proceeding in exactly the same manner one may easily prove that

$$S_{1/m} = \text{Lim}_{n\to\infty}[(1)^{1/m} + (2)^{1/m} + (3)^{1/m} \cdots + (n)^{1/m}]/n^{1+1/m}$$
$$= m/(m+1)$$

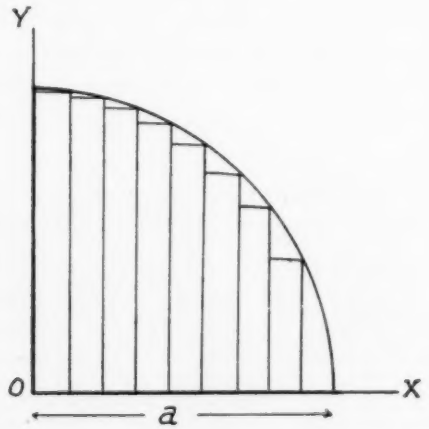
by making use of the general result in 4. Thus it is shown that the formula

$$S_m = \text{Lim}_{n\to\infty}[(1)^m + (2)^m + (3)^m + \cdots + (n)^m]/n^{m+1}$$
$$= 1/(m+1),$$

holds not only for n a positive integer, but also for $m = 1/k$ when k is any positive integer. Hence the way is open for finding, by algebraic procedure areas bounded by the coordinate axes and curves whose equations are of the form

$$y = a + bx + cx^2 + \cdots + lx^m + b^1x^{1/k} + c^1x^{1/(k+1)} + \cdots + l^1x^{1/(k+r)},$$

in which $a, b, \cdots, l, b^1, c^1, \cdots, l^1$ are real coefficients and m, k and r are positive integers. The reader acquainted with the calculus will recognize that what has been done is, in essence, an algebraic-geometric proof that the formula for integrating $x^m dx$, which represents an area, holds for m any positive integer or its reciprocal.



6. To evaluate

$$S = \text{Lim}_{n\to\infty}[(n^2-1)^{1/2} + (n^2-4)^{1/2} + (n^2-9)^{1/2} \cdots (n^2-n^2)^{1/2}]/n^2.$$

Construct one fourth of the circle whose equation is $x^2 + y^2 = a^2$.

Draw line segments $y_1, y_2, y_3, \cdots, y_n$ parallel to the y-axis dividing the radius into n equal parts. Construct rectangles as previously and let A^1 be the sum of their areas, then

$$A^1 = (a/n)[y_1 + y_2 + y_3 + \cdots + y_n]$$
$$= a^2[(n^2 - 1)^{1/2} + (n^2 - 4) + (n^2 - 9)^{1/2} + \cdots + (n^2 - n^2)^{1/2}]/n^2.$$

Let A be the area of one fourth of the circle; then

$$A = \pi a^2/4 = \operatorname{Lim}_{n\to\infty} A^1$$
$$= \operatorname{Lim} a^2[(n^2 - 1)^{1/2} + (n^2 - 4)^{1/2} + (n^2 - 9)^{1/2} + \cdots + (n^2 - n^2)^{1/2}]/n^2$$
$$= a^2S,$$

whence $S = \pi/4$.

7. To find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Construct rectangles in one fourth of the ellipse as in the circle in the last section. Let their bases be of lengths $y_1, y_2, y_3, \cdots, y_n$. Then $y_1 = b[1 - (x_1/a)^2]^{1/2} = (b/a)[1 - (1/n)^2]^{1/2}$; $y_2 = (b/a)[1 - (2/n)^2]^{1/2}$; $y_3 = (b/a)[1 - (3/n)^2]^{1/2}$; $\cdots$; $y_n = (b/a)[1 - (n/n)^2]^{1/2}$. Now A^1, the sum of the areas of the rectangles, each of altitude a/n is $A^1 = (a/n)[y_1 + y_2 + y_3 + \cdots + y_n]$ $= ab[(n^2 - 1)^{1/2} + (n^2 - 4)^{1/2} + (n^2 - 9)^{1/2} + \cdots + (n^2 - n^2)^{1/2}]/n^2$ and A, one fourth of the area of the ellipse is $A = \operatorname{Lim}_{n\to\infty} A^1$ $= ab \operatorname{Lim}_{n\to\infty}[(n^2 - 1)^{1/2} + (n^2 - 4)^{1/2} + (n^2 - 9)^{1/2} + \cdots + (n^2 - n^2)^{1/2}]/n^2 = abS = \pi ab/4$; whence the area of the ellipse is πab.

It will be evident to the reader that much else can be done by this method. For example the areas of many curves whose equations are given in algebraic form in polar coordinates may be obtained by finding the limit of the sum of the areas of adjacent circular sectors inscribed in the curve with their vertices at the pole. This field and others that may suggest themselves will be left for exploration by the interested reader.

HISTORICAL ACCOUNT OF ORIGIN AND GROWTH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

By C. M. AUSTIN

Oak Park High School, Oak Park, Illinois

The winter meeting of the National Education Association was held in Chicago in February, 1919. In the speeches and papers of that meeting, there were many sharp criticisms of mathematics. There were, however, no programs for the friendly discussion of content and method. There were no meetings provided where constructive criticism could be offered and reorganization attempted.

Although the teachers of English had been organized for ten years and were successfully publishing the English journal, the teachers of mathematics did not have any nation-wide organization or journal devoted wholly to secondary mathematics. Several local associations existed, chief of which were the Association of Teachers of Mathematics of the Middle States and Maryland, the New England Association of Mathematics Teachers and the Central Association of Science and Mathematics Teachers.

College teachers of mathematics had two flourishing national organizations with the *American Mathematical Monthly* as their official journal. They were also undertaking the reorganization of secondary mathematics under a grant of money from the General Education Board of New York City.

This task should have been done by the secondary teachers themselves, if they had possessed an organization. The college teachers were able to do it because they had an incorporated national organization and were held responsible for the administration of the fund.

This state of affairs was brought to the attention of the members of the Chicago Mathematics Club at its meeting in March, 1919. A committee consisting of M. J. Newell, E. R. Breslich and C. M. Austin was appointed to correspond with teachers

in other sections in order to ascertain their opinion about the formation of a national organization.

A circular letter was addressed to about one hundred representative teachers of secondary mathematics in various parts of the country. A large majority of the replies favored the project.

Encouraged by the favorable and, in many cases, enthusiastic responses, the committee went ahead with the work. In the autumn of 1919, a call was issued for a meeting to be held at Cleveland in February, 1920, at the time of the annual meeting of the Department of Superintendence of the N. E. A.

February 24, 1920, was the date appointed for the Cleveland meeting. On that day one hundred twenty-seven enthusiastic teachers of mathematics, representing twenty states and as many local organizations, assembled at the Hollenden Hotel. The following announcement which appeared in the February, 1920, issue of *School Science and Mathematics* clearly shows the purpose and program of this meeting.

NOTICE! NOTICE!

ON TO CLEVELAND!

Mathematics teachers are to have a place on the program at the Cleveland meeting of the N. E. A., which meets February 23–28, 1920.

Professor C. N. Moore of the University of Cincinnati will speak on "The Claims of Mathematics as a Factor in Education."

Besides the above address there will be an all day session of mathematics teachers on Tuesday, February 24, held in the Hollenden Hotel for the purpose of organizing A National Council of Mathematics Teachers.

This movement has been endorsed by The Association of Mathematics Teachers of the Middle States and Maryland, and by the Mathematics Section of the Central Association of Mathematics and Science Teachers. The following tentative program has been arranged:

MORNING SESSION, 9:00 A.M.

J. A. Foberg, Crane Junior College, Chicago, presiding.

Organization of a National Council of Mathematics Teachers.

1. Suggestions from The National Council of English Teachers, James F. Hosic, President.
2. Submission of Plans for Organization.
   (*a*) By The Association of Teachers of Mathematics of the Middle States and Maryland.

(*b*) By The Central Association of Science and Mathematics Teachers.

(*c*) By the New England Association of Mathematics Teachers.

3. Discussion.
4. Appointment of Organization Committee to report at afternoon session.
5. Reports of Activities of Various State and City Mathematics Organizations.

AFTERNOON SESSION, 2 P.M.

D. W. Werremeyer, Director of Mathematics in Cleveland Public Schools, presiding.

1. Report of National Committee on Mathematical Requirements, Dr. J. W. Young, Dartmouth.
   Discussion by Eugene R. Smith, Principal Park School, Baltimore, Md., Frank C. Touton, High School Inspector for Wisconsin.
2. Mathematics of the Junior High School, William Betz, Lincoln School, New York.
   Discussion led by Miss Marie Gugle, Assistant Supt. Schools, Columbus, Ohio.

Whether or not a national organization should be formed was thoroughly discussed at the morning session. As I recall it, the opinion of the group was unanimously favorable to the project. Some expressed themselves as favoring the movement provided a journal could be secured. Others wished to organize the Council and then settle the question of the *journal* afterwards. At the end of the debate, the group decided to go ahead and perfect the organization. A scheme of organization was presented by the Chicago delegation. The following named persons were appointed as a committee to bring in a constitution at the afternoon session:

H. O. Rugg, New York City, Chairman,
E. R. Smith, Baltimore, Md.,
C. M. Austin, Oak Park, Ill.,
Marie Gugle, Columbus, Ohio,
J. A. Foberg, Chicago, Ill.

The committee submitted the following constitution, which was adopted:

THE CONSTITUTION FOR THE NATIONAL COUNCIL OF MATHEMATICS TEACHERS, WHICH WAS ADOPTED AT THE MEETING IN CLEVELAND, OHIO, FEBRUARY 24, 1920

I. Name

This organization shall be known as the National Council of Teachers of Mathematics,— those who are primarily interested in elementary and secondary mathematics.

II. Objects

The purposes of this organization shall be,

1. To secure a greater degree of cooperation and solidarity among the teachers of mathematics.
   (*a*) To provide for wide publicity of important reports and addresses related to mathematics and the teaching of mathematics, through an official organ and other publications.
   (*b*) To vitalize and coordinate the work of the many organizations of mathematics teachers throughout the country.
2. To bring the interest of mathematics to the attention and consideration of the educational world.

III. Time and Place of Meeting

Regular meetings of the Council shall be held annually in connection with the meeting of the Department of Superintendence of the National Education Association.

IV. Membership

Membership in the Council shall be of two kinds: individual and collective.

All persons who are in sympathy with the work of the Council shall be eligible to individual membership.

All organizations of mathematics teachers shall be eligible to collective membership.

The annual dues of individual members shall be two dollars.

The minimum dues for collective membership shall be three dollars for each organization of less than fifty. For each additional hundred members, or fraction thereof, the dues shall be an additional five dollars.

V. Management of the Council

1. *Officers*

The officers of the Council shall be a President, a Vice-President, and a Secretary-Treasurer. Their duties shall be those

commonly pertaining to these offices. The President and Vice-President shall be elected for a term of one year. The Secretary-Treasurer shall be elected for a term of three years.

2. *Executive Committee*

There shall be an Executive Committee of nine members three of whom shall be the officers. The remaining six shall be elected, two each year, to hold office for a term of three years; except that at the first meeting, two shall be elected for one year, two for two years, and two for three years.

The Executive Committee shall manage the business of the Council, authorize the appointment of committees, and fill vacancies in office.

3. *Nominations*

At each annual meeting the President shall appoint a nominating committee of three members who shall report nominations for officers and for members of the executive committee.

IV. Amendments

This constitution may be amended by a two-thirds vote of the members present at any regular meeting.

Committee on organization,
H. O. Rugg, *Chairman*,
E. R. Smith,
C. M. Austin,
Marie Gugle,
J. A. Foberg.

A second committee headed by Mr. F. C. Touton, of Wisconsin, submitted the following named persons as nominees for the first officers of the National Council:

President, C. M. Austin, Oak Park, Ill.
Vice-President, H. O. Rugg, New York City
Secretary-Treasurer, J. A. Foberg, Chicago, Ill.
Executive Committee
For three years:
Marie Gugle, Columbus, Ohio
Jonathan Rorer, Philadelphia, Pa.
For two years:
Harry Wheeler, Worcester, Mass.
W. A. Austin, Fresno, Cal.
For one year:
W. D. Reeve, Minneapolis, Minn.
W. D. Beck, Iowa City, Iowa.

Quite a full account of this meeting appeared in the September, 1920, issue of the MATHEMATICS TEACHER.

On February 25, 1920, the Executive Committee held its first meeting. Mr. J. R. Clark was unanimously chosen as Editor and Mr. J. A. Foberg as Business Manager of the official journal. Each member present pledged $50.00 to the Council for publicity and other expenses. They were never asked to redeem their pledges. There was discussion on the question of the official journal. Some favored the establishment of a new magazine; others wanted to make an effort to secure a magazine published by the Association of Teachers of Mathematics of the Middle States and Maryland and the New England Association of Mathematics Teachers and known as the MATHEMATICS TEACHER. The committee decided to follow the latter course.

During the summer and autumn of 1920, the committee worked on this proposition. Mr. Austin had two conferences with Dr. W. H. Metzler, then Editor of the MATHEMATICS TEACHER. The proposed action was presented to the two eastern associations at their regular meetings. They recorded a favorable vote and the journal was transferred to the ownership and managership of The National Council of Teachers of Mathematics. This action on the part of Dr. Metzler and the two eastern associations is regarded by the writer as most magnanimous. They were surely inspired by the most unselfish motives. They sacrificed their personal rights and desires for the larger good. They saw the immense benefit to be gained in the service that the magazine could render under the ownership of the Council. Instead of bringing four numbers to one thousand teachers each year, it now brings eight numbers to over four thousand each year. It was stipulated only that the transfer should work no hardship upon the old subscribers to the magazine. This was guaranteed by the officers of the National Council. Each old subscriber was to receive just as many copies of the magazine under the new management as he was entitled to receive under the old.

All of these arrangements were completed early in the autumn of 1920. So, in January, 1921, the first number of the MATHEMATICS TEACHER was brought out by The National Council. The editorial board consisted of

J. R. Clark, *Editor-in-Chief*
Eugene R. Smith, *Associate Editor*
Alfred Davis
Harry D. Gaylord
Marie Gugle
John W. Young
J. A. Foberg, *Business Manager*

Under the leadership of Mr. Clark, the journal continually gained in popularity. It promotes good scholarship and inspires all teachers with a desire to do better work in the classroom.

One of the outstanding features of the National Council has been the unselfish service given by members and officers. For three years Mr. Clark even donated the time needed to edit the journal.

At the annual meeting in Chicago in 1924, a committee consisting of Miss Marie Gugle, Miss Mabel Sykes, Miss Orpha Worden, Miss Florence Bixby, J. A. Foberg, W. D. Reeve, Raleigh Schorling, C. M. Austin, E. H. Taylor and E. W. Schreiber recommended to the Council "that the Executive Committee be authorized and directed to enter into a legal contract with John R. Clark by which he agrees to edit and manage the Mathematics Teacher for a period of five years on the following conditions:

"1. The subscription price of Mathematics Teacher including membership in the Council is to be two dollars, of which the Editor-Manager shall receive one dollar seventy-five cents and the Council shall receive twenty-five cents.

"2. Each number of Mathematics Teacher shall contain a minimum of 62 pages of printed matter exclusive of advertisements.

"3. There shall be eight issues of the Magazine each year.

"4. The proprietary interest in the Mathematics Teacher shall continue to rest in the National Council of Teachers of Mathematics." This report was adopted.

Mr. Clark continued to edit and manage the Mathematics Teacher under this contract until February, 1927. Then feeling that the work was too heavy to be done alone, he asked that W. D. Reeve, of Teachers College, Columbia University, be chosen as Co-Editor and Business Manager. This action was author-

ized by the Executive Committee at Dallas, Texas, on February 25, 1927. A new contract for five years was made between the officers of the Council and Mr. Clark and Mr. Reeve. The terms of the new contract are the same as the terms of the one made in 1924. It was also decided that Mr. Clark and Mr. Reeve should get out the yearbooks.

The Constitution adopted in 1921 has been twice amended. In 1922, Section IV was amended by striking out all reference to collective membership. In 1925, Section V, Article 3, was amended to read as follows: "The Executive Committee acting as a nominating committee, shall nominate two candidates for each office to be filled and these nominations and ballots shall be published in the MATHEMATICS TEACHER in December and a vote shall be taken by mail. The election shall be closed on February 1st and ballots shall be counted by the Executive Committee at the annual meeting. Those receiving a majority of the votes shall be declared elected. Only subscribers to the MATHEMATICS TEACHER shall be entitled to vote." This method has not proved altogether successful, as only a very small number avail themselves of the opportunity to vote.

At the meeting last year in Dallas it was decided by unanimous vote that the old constitution was outgrown and a committee consisting of H. E. Slaught, C. M. Austin and E. W. Schreiber was authorized to revise it. The committee, after some deliberation, decided to draft a new instrument for the government of the Council. The new Constitution and By-Laws, as printed in the January, 1928, issue of the MATHEMATICS TEACHER, was presented and adopted at the meeting of the Council at Boston in February, 1928.

The meeting at Cincinnati was a notable one, not only because of the fine program rendered under the leadership of President Schorling, but also because the first yearbook was authorized. It had been the custom of many educational societies to issue yearbooks annually. In these books were published the proceedings and papers of the annual meeting or significant studies and reports made by members of the society. With these facts in mind, it was proposed at the 1925 meeting that the National Council publish a yearbook. After some discussion the proposal was adopted. A committee, consisting of F. C. Touton, Harry English, W. C. Eells, Wm. E. Betz and C. M. Austin, was ap-

pointed to prepare and publish the book. "Twenty-five Years of Progress in the Teaching of Mathematics in the United States" was the title chosen. The book was ready for the Washington meeting in February, 1926. Thirty-two hundred copies were printed. Only a few copies now remain unsold. From this doubtful experiment, a net profit of almost two thousand dollars has come into the Council treasury.

A second yearbook on "Curriculum Problems in Teaching Mathematics" was published in 1927. The committee in charge consisted of C. N. Stokes, C. B. Marquand and W. D. Reeve. Up to date, about four thousand copies have been sold.

A third yearbook on "Significant Topics in the Teaching of Mathematics," as well as the second yearbook, is now on sale at the Bureau of Publications, Teachers College, 525 West 120th Street, New York City. These yearbooks, together with the files of the MATHEMATICS TEACHER, form an almost invaluable library on the teaching of mathematics.

Counting the meeting for organization, eight annual meetings have been held—two at Cleveland, two at Chicago, one each at Atlantic City, Washington, Cincinnati and Dallas. Considering the great distances to travel, the attendance has been exceptionally large—averaging about one hundred and seventy-five persons. Ten to twenty states were represented at each meeting. These national meetings have been valuable not only for carrying on the work of the National Council, but also because they have been a great inspiration to the local teachers of the community. The meeting at Dallas, Texas, in 1927 well illustrates the point. Two hundred teachers from fourteen states attended. Oklahoma and Texas furnished the largest number. No meeting like this had ever come to that part of the country. The Texas teachers, with one accord, testified to the wonderful impetus and inspiration given to their work by the National Council meeting.

While all the things desired have not yet come to pass, while only a small part of the teachers of Mathematics in the United States have been reached and influenced by our organization and its publications, yet the writer of this account feels that the many things already accomplished amply justify the faith of those who founded the National Council and who are still working to give it a larger place in the educational world.

CHART SHOWING THE OFFICERS OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

	1921	1922
Pres.	C. M. Austin	J. H. Minnick
V.-Pres.	H. O. Rugg	E. H. Taylor
Sec.-Treas.	J. A. Foberg	J. A. Foberg
Ex. Committee	W. D. Reeve	Harry Wheeler
	W. D. Beck	W. A. Austin
2 years	Harry Wheeler	Marie Gugle
	W. A. Austin	Jonathan Rorer
3 years	Marie Gugle	Orpha E. Worden
	Jonathan Rorer	C. M. Austin
Annual Meeting at	Atlantic City	Chicago

	1923	1924
Pres.	J. H. Minnick	J. H. Minnick
V.-Pres.	Eula Weeks	Mabel Sykes
Sec.-Treas.	J. A. Foberg	J. A. Foberg
Ex. Committee	Marie Gugle	Orpha E. Worden
	Jonathan Rorer	C. M. Austin
2 years	Orpha E. Worden	Gertrude Allen
	C. M. Austin	Mr. Rankin
3 years	Gertrude Allen	Eula Weeks
	Mr. Rankin	W. C. Eells
Annual Meeting at	Cleveland	Chicago

	1925	1926
Pres.	Raleigh Schorling	Raleigh Schorling
V.-Pres.	Florence Bixby	Winnie Daley
Sec.-Treas.	J. A. Foberg	J. A. Foberg
Ex. Committee	Gertrude Allen	Eula Weeks
	Mr. Rankin	W. C. Eells
2 years	Eula Weeks	Orpha E. Worden
	W. C. Eells	C. M. Austin
3 years	Orpha E. Worden	Harry English
	C. M. Austin	Harry Barber
Annual Meeting at	Washington	Cincinnati

	1927	1928
Pres.	Marie Gugle	Marie Gugle
V.-Pres.	W. W. Hart	C. M. Austin
Sec.-Treas.	J. A. Foberg	J. A. Foberg
Ex. Committee	Orpha E. Worden	Harry English
	C. M. Austin	Harry Barber
2 years	Harry English	W. D. Reeve
	Harry Barber	F. C. Touton
3 years	W. D. Reeve	Vera Sanford
	F. C. Touton	Wm. Betz
Annual Meeting at	Dallas	Boston

THE LESSON OF DEPENDENCE

By DAVID EUGENE SMITH

Professor Emeritus of Mathematics, Teachers College, Columbia University

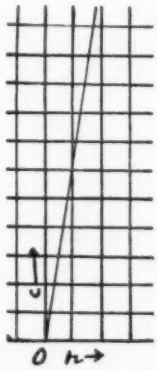
Notion of Dependence.—A child in the seventh school year learns how to find the circumference of a circle. He sees no particular reason for learning it; he does not use it, nor is it used in his home. If he enters certain types of industry, he will need it, but probably not otherwise. If his teacher connects it with the speedometer on his father's car, it immediately has some interest; and if it is similarly related to a bicycle, the interest is increased and the way is open to other applications that give reality to the abstract. If the law is expressed algebraically, he learns that $c = 2\pi r$, or $c = 6.28r$ approximately, and the concrete again returns to the abstract. If he is told that this is merely a piece of shorthand, his interest is again awakened—awakened by the mystery of a name—he is writing a kind of shorthand, a universal language which is the same in France and Germany and Japan and all the civilized countries of the world.

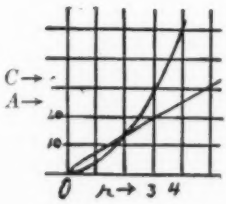
But there is, in the formula $c = 6.28r$, something greater than all this. For any particular radius; that is, for any particular value of r, c has one fixed value. If r changes, so must c change; if r is doubled, c is also doubled; as r increases, c increases; and when r shrinks in size, c shrinks at the same rate. If $r = 0$, $c = 0$; and if $r = 1$, $c = 6.28$. If we draw a graph, as shown below on the left, OP represents the relation between c and r.

We may therefore say that *c depends upon r for its value.* Hence, we may speak of r as a *variable*, and c is *a dependent variable.*

If we find the area of a circle, we may write the shorthand statement $A = \pi r^2$, or $A = 3.14r^2$. Here again, r is a variable, an *independent variable* as we say, while A is a dependent variable. The graph is not the same as in the case of $c = 6.28r$. It was then a straight line, a linear graph; now it is a curve.

If we plot both graphs on the same axes as shown above at the right, but for convenience in not having the figure too high we





use a unit ten times as long on the r axis as on the y axis, we see that the area at first increases more slowly than the circumference; but that after r passes the value 2, the area increases with very great rapidity, the curve rising quite abruptly.

Some forms of dependence may, therefore, be thought of as mild while others are of a stronger and more assertive nature.

Mathematics, like life, is full of examples of dependence. If we take the formula for the area of the entire surface of a cube of edge e, we have

$$A = 6e^2;$$

and if we consider the volume, we have

$$V = e^3.$$

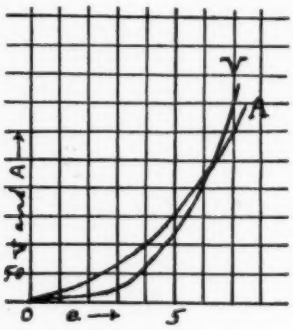
If we draw the graph, we shall find that, at first, V increases less rapidly than A; but after $e = 6$ we see that it increases with much greater rapidity. In each case, however, we see that the values of V are tied up to the values of e and of A and that the union is eternal.

Similar illustrations may be found in arithmetic, in mechanics, in all branches of science and in all the walks of life. For example,

In computing interest, $i = prt$;
In compound interest, $A = p(1 + r)^n$;
In a trapezoid, $A = \frac{1}{2}h(B + b)$,

and so on as far as we may wish to search out examples.

But dependence in this life is never one-sided. You depend upon your friend for good company, for encouragement, for honest counsel, and for help; but in each case your friend also



depends upon you. This is the essence of friendship, that the dependence is mutual. Sometimes you act freely and your act influences your friend; sometimes this is reversed, and he acts independently and you are influenced by him. There is no such thing as a perfectly independent variable in your action or in his, nor is there such a thing as perfect independence.

If we consider the formula

$$A = \pi r^2,$$

where π, as usual, stands for approximately 3.14, we see that if r acts independently, then A is dependent upon the result; but if A acts independently, then r depends upon the result.

Similarly, in our lives as in mathematics, either friend may act the independent variable, and the other is influenced by that act.

If we wish to appear very learned, we look at a formula like

$$c = 2\pi r,$$

and we say that c is a *function* of r, which is only a rather high-sounding way of saying that c depends upon r for its value. Such terms have their value, but in the early stages this value is negative.

Dependence in Life.—Mathematics is a kind of science of dependence. The cost of a piece of cloth depends upon the length,

upon the quality, upon the width, and so on—but there is always dependence of some kind. Similarly, all costs, all expenses, all joy, all sorrow, all weight, all volume, all success—everything depends upon something else. We may seek for freedom, but we shall never find it; our comfort depends upon the price of coal, our health upon a colony of microbes, our success in school upon our power of will—the will to conquer, and so on through every act or thought or sensation. The success of civilization depends upon the cooperation of all people for the common good, upon their "doing the square thing" by their neighbors, upon their education, upon their will to conquer nature and to sacrifice for others. Our lasting pleasures depend upon success in life, our success depends upon our efforts and upon our inborn qualities, and so on through all that enters into our life here and hereafter.

A triangle in ordinary space, if given life, could not change the sum of its angles; its very existence depends upon the inexorable law that this sum is 180°. The triangle may change its shape, but its area depends upon the lengths of all its sides, and the length of any side depends upon the lengths of the other two sides and the size of the opposite angle.

If we say, "We are free to act as we will," we might as well be circles and say "Let our several radii change as they wish, we shall make our areas what we will." If we say, "I will utterly perish when I die, body and soul," we may as confidently say, "When a line has shrunk to zero, it can shrink no more," or it is impossible to conceive of or to picture a negative line or an imaginary number.

When a comet comes out of the Great Unknown, swings with incredible speed around the sun, and returns whence it came, it may be imagined as thinking that it is independent in its path; but to us who see it and who know the everlasting laws which hold it to its parabolic course, it is evident that not for a single instant has it independence of action. Mr. Sperry, whose gyroscopes are used for navigating as well as stabilizing great ships and swift airplanes and which have made his name known throughout the world, once told me of meeting Mussolini. The latter, knowing of a ship that Mr. Sperry had stabilized for the Japanese Navy, remarked, in course of their conversation, "I cannot see how you could keep a great warship steady in heavy

seas unless you reached out and laid hold of something as solid as the Rock of Gibraltar." "You are right, your Excellency," replied Mr. Sperry, "that is just what we do—we reach out and lay hold of a great eternal law." He then proceeded to demonstrate to Il Duce the astonishing magnitude of this law illustrated even in a minute model which he took from his pocket.

What does the function concept, the idea of dependence, mean to us? What does it tell us of our rather childish claims to independent action? If the comet moves unaware of the eternal law that controls its motion, what about our movements, our thoughts, and our actions? Is "obedience nobler than freedom"? Is freedom not always obedience? Are we ourselves anything but dependent variables, points on some vast graph of whose law we are unconscious? And if we are so dependent, what are the independent variables upon which we depend? Does our free will help us to select a certain few of these; and if so, are we making the selection that, in the infinitely long run, brings success?

What has mathematics to do with our life now and hereafter? Might we not better ask, "What has it not to do?"

THE NATIONAL COUNCIL YEARBOOKS

The attention of our subscribers is again called to the yearbooks of the National Council. The First Yearbook on "A General Survey of Progress" may be secured for $1.10 from C. M. Austin, Oak Park High School, Oak Park, Illinois. The Second Yearbook on "Curriculum Problems in Teaching Mathematics" may be secured from the Bureau of Publications, Teachers College, 525 West 120th Street, New York City. The unbound volume is $1.25, the bound $1.50. The Third Yearbook on "Selected Topics in the Teaching of Mathematics" will also be published by the Bureau of Publications, Teachers College, and can be secured on and after February 20th, for $1.75 (Bound Volume). Send in your orders before the supplies are exhausted.

CONTRIBUTIONS OF MATHEMATICS TO MODERN LIFE [1]

By DR. C. C. CAMP

The University of Nebraska, Lincoln, Nebraska

The English mathematician and philosopher, Bertrand Russell, has defined mathematics as the subject in which we never know what we are talking about nor whether what we are saying is true. Perhaps this definition will appeal to some of you as fitting in well with your experience!

It is not my purpose this evening to induce you all to specialize in mathematics. It might not be the best thing for you—nor for the science of mathematics! One of the chief elements of an education consists in the ability to appreciate what individuals in other lines are doing and what they have contributed as a heritage to us from the past. I shall have fulfilled my aim, then, if I can give you an appreciation of some of the chief contributions of mathematics to your life and mine.

Some people think of a mathematician as one who can add up a column of figures without making an error. The fact is that so few mathematicians can do this that a calculating machine has been invented to do it for them!

Aristotle defined mathematics as the "science of quantity." Other individuals have thought of it as "intelligence in its purest form," or the "theory of pure forms including their relationships and properties," or as "pure logic," or even as a "fine art." Be that as it may, it is clear that any consideration involving numbers, magnitudes, exact relationships, operations, or order is aided by the use of mathematics.

The leading characteristic of man is his power to think consistently. The existence of mathematics which has persisted in almost the same form for many centuries (the mathematics of the ancient Greeks is valid to-day) gives the human mind the most cogent reason yet advanced for believing in its own power

[1] Lecture delivered before the Freshmen of the Arts and Science College, University of Nebraska, January 15 and 16, 1928.

to think consistently and in the essential accuracy of its careful processes. Surely this is a distinct contribution to our life.

"Mathematics is essentially easier than the other branches of learning. The appearance of greater difficulty, which has deceived most people, is due to the fact that mathematics, being the oldest science, is relatively further advanced than every other subject." Sometimes the difficulties of mathematics may be overcome, if one is resourceful. A certain colored clergyman was conducting an evening service in a chapel formerly occupied by an Anglican church. He discovered a hymnal there and, looking through, found a hymn suited to his subject, but at the top instead of the ordinary numerals he saw the letters CXIX. Being unfamiliar with Roman numerals, he was at a loss how to announce the hymn. He read the verses through with signs of uneasiness and then he reread the first stanza, which didn't seem to help him nor his audience. Then he straightened himself, and said with dignity, "Brethren, let us sing the Skeesix hymn!"

"Since it pushes its conquests out in many directions, is frequently annexing new domains, never yields up what it has once attained and remains youthful in its spirit of conquest, mathematics will become, if it has not already, the most extensive scientific doctrine in the whole range of knowledge."

A moment ago I mentioned a calculating machine. Such machines are now run by electricity and will multiply and divide as well as add. We must admit that this machine and, in general, all precise instruments and machinery have been made possible by modern mathematics. You ask why. Because mechanical engineering is a branch of applied mathematics and by virtue of it all precision instruments are designed and turned out.

Consider as a second illustration the compound microscope. This machine involves lenses studied in Optics, a branch of Physics, which is another field of applied mathematics. The surfaces of the lenses are geometric figures studied by the mathematician.

You have studied arithmetic, but how many of you can find the edge of a cube whose volume is given? This can be done by algebra which you are somewhat acquainted with. After a year or two of algebra how many of you would define it as the art of

avoiding tedious numerical calculations? But this is the correct view from the practical standpoint. Let me illustrate. A man bought a house and lot for $750. (This was in a small town and before the World War!) He agreed to pay $50 down and $10 each month plus accrued interest at 7 percent until it was paid for. What actually happened was that he paid the first monthly payment in advance (since no interest was then due), and at the beginning of the next month $15. He was poor at figures but found that this would cover the interest due. So he continued paying $15 each month for 43 months. Then he became uneasy, not knowing how many payments were necessary, and presented his problem to a friend of mine who was good at arithmetic, asking him to work out the number of payments and the amount of the last one. My friend worked the whole problem out by arithmetic, taking 7/12 percent of the unpaid balances and deducting $15 each time, and the calculations required about three hours. Contrast to this what may be accomplished by the use of algebra and modern annuity tables. One glance at the table shows that ten more payments are necessary. In less than two minutes the amount of the last payment, $10.74, may be found. Thus algebra reduces three hours to two minutes. The slide rule carried by all engineers is an adaptation from algebra. By it they multiply, divide, do square and cube roots, etc., with three-digit accuracy. The same is done by logarithmic tables to as many as 20 digits when greater accuracy is required.

I wonder how many of you have heard of the Moffat tunnel? By the use of trigonometry, or the study of triangles, men were enabled to cut through the mountains west of Denver from both sides with practically no deviation at the meeting point. A system of triangulation makes possible geodesy or the accurate charting of the earth's surface on maps. Mountain peaks are measured which have not been scaled. Without trigonometry it would be impossible for navigators to cross the seas with any safety.

In astronomy mathematics has helped determine the orbits of the planets as ellipses with the sun at one focus. By use of the spectroscope the velocities of the stars and of our own sun have been found. Eclipses are accurately predicted. But there is still work for the mathematician. The problem of three bodies, *i.e.*, to determine their motions when each is attracted by the

other two, has not been completely solved although it has been done under certain special conditions. The astronomer is therefore handicapped and is waiting for the mathematicians to solve this problem. Perhaps here is a chance for some of you. This problem will never be solved by a man like Pat. Pat and another Irishman had a day off, so they thought they'd go to Atlantic City to spend the day. They were walking along the famous Board Walk by the sea enjoying the scenery when one of the Irishmen said to the other, "Pat, would yer like to follow the sea always?"

"Shure, an' that Oi would," replied Pat, "if Oi could go the whole distance on the boardwalk."

By the Einstein theory of relativity physics has been reduced to four-dimensional geometry. And by the modern electron theory the dream of alchemists, or the changing of one metal into another, is coming true. Not only that but the whole science of Chemistry is being reduced to pure dynamics, a branch of applied mathematics.

And now we come to the man of the hour, Lindbergh. How was it possible for him to make his flight to Europe? Because of his wonderful gyroscopic compass. This instrument involves an "awful lot" of mathematics. By it he was enabled at any moment of the dark foggy night to tell in what direction he was going and whether he was following his course. Mathematics also properly applied made possible the "Spirit of St. Louis" so dear to him.

The steam engine involves thermodynamics, which is almost pure mathematics; the dynamo is a product of electrical engineering, which is highly mathematical; the telephone is another; even modern surgery has been made possible by means of mathematics. You say that mathematics alone didn't produce all these things. I agree that other sciences contributed along with mathematics. The point I wish to make is that the part contributed by mathematics is an essential element.

There is a close relationship between the production of musical sounds and mathematics, which has contributed toward better instruments and better acoustics in auditoriums.

In the World War the German long-range gun, Big Bertha, was located by the mathematical theory of sound propagation. The form of blunt-nosed shells for maximum range, used in our

guns, was determined by a branch of mathematics known as the calculus of variations. This also aids us in devising the forms of airplane wings and propellers.

Perhaps the greatest marvel of science is that connected with the square root of minus one. You have met this number in algebra and it was called an imaginary. Even mathematicians had a century's struggle with it before they admitted it as a number at all. The marvel is that highly mathematical theory involving this number made possible and directed the invention of wireless telegraphy. Without this it is probable that the radio would never have existed. Mathematical research very definitely made possible long distance telephoning, and is now engaged in the improvement of wireless telephony and television.

There were three famous problems which worried the ancient Greeks: the trisection of an angle, the duplication of the cube, and the squaring of the circle. These were to be solved by the use of the compass and straight edge. Apollo was thought to be angry and to appease him an altar was to be constructed in the form of a cube, with a volume just twice that of a given cubical altar. The story of the third problem, or the finding of a square equal in area to a given circle, covers a period of 24 centuries, which shows the distinct progress of the science. In 1882 all three of these problems had been proved impossible. Sometimes it is important to show that a thing cannot be done. But as early as the sixth century the Chinese had determined π to seven decimals as 3.1415926+, also the fractional equivalent of its first seven digits, 355/113. Much later Shanks of England entered into a wager and calculated π to 707 decimals. But he could not discover a law by which the next figure could be written down.

Descartes, the founder of modern science, was also the inventor of analytical geometry. This may be called the beginning of higher mathematics, by which we have a tremendous advantage over the ancient Greeks.

You have had in plane geometry the length of a circular arc and the areas of certain figures. About 1666 Isaac Newton invented a method he called fluxions, a branch of mathematical analysis, which we now call calculus. By this we are enabled to find the length of any arc, the area under any curve, the volume of any solid bounded by surfaces, whose areas can also

be found. By it velocities and accelerations are computed, centers of gravity are located, likewise the maximum and minimum of quantities which vary. Engineers use this to calculate the maximum strains on a proposed bridge, then use a factor of safety. Huge bridges and skyscrapers would be unsafe except for the mathematical calculations of the engineer. The Greeks were unable to estimate the crushing strength of columns and for that reason built them over-large.

The so-called exact sciences are exact because of their mathematical content. Moreover, the so-called non-exact sciences such as sociology, psychology, economics, botany, zoology and geology are becoming more and more nearly exact by the application of mathematical reasoning—in particular that of the theory of mathematical statistics. I am interested in statistics especially since I am planning to offer a course in statistics next fall. The prerequisite will be one year of college mathematics. Later I expect to give more advanced courses if a sufficient number of you become interested.

Let me say that although "political arithmetic," as the subject was originally called, began a long time ago when men were counted for military duty, it is only recently that the science of statistics has been applied to almost every phase of human life.

One of the obvious applications of statistics occurs in the theory of insurance, or actuarial science. Life insurance companies of long standing went to the wall some sixty years ago because of the lack of reserves, which they were unable then accurately to estimate. Now we have a so-called experience mortality table showing the number of deaths during successive years of a group of 100,000 people. Reserves can therefore be computed accurately nowadays and insurance companies are consequently very safe.

Quetelet, the founder of modern statistics, by using the so-called "laws of large numbers" developed the idea of the "average man."

Without statistics, sociology, or the nature of human society, could never become known, but the method is applicable to all the sciences. It is now being taken up by engineers. There are economic statistics, in which business cycles and index numbers are considered; biometry or biological statistics, developed by Sir

Francis Galton (1822–1911) and Karl Pearson of London; and psychological and educational statistics.

At one time the politicians so misused and misinterpreted statistical data that people were wont to say, "You can prove anything by statistics!" A certain college dean always got angry when someone began to quote statistics. He said there were just three kinds of lies: the white lie, the (I'll say black lie although that is not what he called it), and statistics! Nowadays, with the more accurate study of mathematical statistics, such statements are no longer true.

Suppose that a shoe manufacturer wants to know the distribution of sizes of shoes; *e.g.*, what proportion of men wear No. 8 He would like to know the distribution of foot sizes for all the people in our population. Needless to say, only a sample may be measured and tabulated. If a sufficiently large number of feet were measured, a curve approximating the so-called normal probability curve or curve of chance would be obtained. If properly drawn, such a curve would give the proportion of people who should wear No. 8, as well as the other sizes. In order to describe such a distribution or frequency curve we need four quantities: the arithmetic mean, the standard deviation, the third and fourth moments.

What can we say of the hundred million people in this country from a study of one or more fairly large random samples? Will several samples lead to the same inference? What is the probable error of the mean and of the other three quantities calculated for the whole from the samples? What can be inferred from a small sample, or several of these? Such questions are of great importance. Here is a field of research, perhaps for some of you.

At the present time there is a great need for statisticians in commerce and industry as well as in actuarial science and government work. The large electric companies have their statisticians. Statistics in the future is bound to be one of the most extensive contributions of mathematics to all phases of human life.

In closing I wish to read a quotation from Professor Keyser of Columbia University. Sometimes the value of a person or thing can be judged by the effect of its absence. He says:

". . . if all these mathematical contributions were by some

strange spiritual cataclysm to be suddenly withdrawn, the life and body of industry and commerce would suddenly collapse as by a paralytic stroke, the now splendid outer tokens of material civilization would quickly perish, and the face of our planet would at once assume the aspect of a ruined and bankrupt world. For such is the amazing utility, such the wealth of by-products, if you please, that come from a science and art that owes its life, its continuity and its power to man's love of intellectual harmony and pleads its inner charm as its sole appropriate justification."

"The relations with which the mathematician deals seem to be a part of the very foundation of the world we live in, so that we have discovered that, if any proposition that holds true of experience is elaborated in accordance with the rules of mathematics, the conclusions thereupon reached will also hold of experience. This fact about our universe, and the additional fact that the quantitative methods of mathematics admit of the utmost accuracy and precision of formulation, explain why in each of the fields we have looked at so far mathematics is so fundamental. It does seem to be true that the more highly developed a science becomes and the more knowledge we gain about the relations between its object, the more its beliefs tend to fall into mathematical methods. So true is it that a science is successful just in so far as it is able to formulate its beliefs mathematically, that many men have naturally come to think that in mathematics is to be found the exemplar of all true knowledge."

Introduction to Reflective Thinking by Columbia Associates in Philosophy (Houghton Mifflin, 1923).

THE GRAPH AS A MEANS OF PICTURING RELATIONSHIPS [1]

By H. C. CHRISTOFFERSON

Teachers College, Columbia University

In the last few years there has been a tremendous wave of progress in the field of secondary school mathematics. The ideas of Perry in England, Tannery in France, Klein in Germany, Moore and Myers in Chicago, followed up and amplified by men like Professor Hedrick of Southern California, Young at Dartmouth, Reeve at Teachers College, Breslich at Chicago, Schorling at Michigan, Clark at New York University and others are beginning to be felt. This is evidenced by the great number of textbooks published during the last few years.

An incomplete list of textbooks for ninth grade mathematics published since 1914 shows that during the last 13½ years there have been published 79 different competing books, and during the last 4½ years 53 books, 13 in the one year 1925, and 19 in the one year 1924. In all cases only the latest edition of any one series was counted. One book company alone is now publishing 10 different competing ninth grade Mathematics books, and another, 8 competing books.

Why this apparently senseless competition, this great number of new books being published? Evidently many people are dissatisfied with the present course and during a lucid moment get an inspiration which changes their course even slightly and eventually results in another textbook. We must then admit that ninth grade Mathematics, call it Algebra, Modern Algebra, General Mathematics, or any one of a dozen other titles used, is undergoing a tremendously vital and soul-stirring evolutionary growth, from which may eventually emerge the perfect ninth grade mathematics course. This course we hope will be as inspiring and fundamental in the training of all American citizens as that of Euclid has been for a selected few in the past.

[1] This paper was read before the meeting of the Mathematics section of the Middle Atlantic States and Maryland Teachers Association at Atlantic City, November 26, 1927.

The best thought seems to indicate that the idea of dependence, of functional relationship, will be the unifying element of the entire six years of secondary school mathematics. It is the most simple, the most fundamental, and the most fruitful in "lead on" possibilities of all mathematical concepts. This is recommended by that almost prophetic National Committee under the leadership of Professor Hedrick, and is now waiting for some genius to take it, to develop it, and to interpret it so that it may be easily used and understood by children.

Not being that genius, the writer purposely sidesteps this major problem in the reorganization of mathematics and will discuss the secondary problem of using graphs to picture functional relationship.

Let us first expose the inadequacies of some present treatments. The sole use that some authors make of graphing beyond statistical graphs is in the solution of simultaneous and quadratic equations. They then immediately solve these equations by the shorter and better algebraic methods which all students ever afterward naturally use. Graphing becomes then merely a blind alley more useless than factoring and as futile as L.C.M. or G.C.F.

One book gives a problem in graphing like this: "Draw a graph showing the cost of oranges at 50 cents a dozen and from the graph determine how much you would pay for 3½ dozen oranges." Can you conceive of any housewife who has a family large enough to require 3½ dozen oranges at one purchase or any merchant selling oranges who would find their cost by graphing?

Another problem: "By graphing determine the interest on $100 for 2½ years at 5 percent." Even neglecting the fact that people never compute interest for such periods of time, no banker or borrower would ever figure interest by using a graph.

These mistakes are not to be lamented, in fact they are a hopeful sign of growth and change. They merely indicate that our mathematics tree is growing rapidly and needs some pruning to hasten the process of selection in the evolution of mathematics.

The first and simplest form of graphing is the graph of statistical data. It is not within the scope of this paper to discuss the commonly known facts about this type of graph. Bar and circle graphs have tremendous value as significant projects fur-

nishing motivated drill in numbers, angle measure, proportion, scale drawing, percentage, and also furnishing purposeful activity for developing habits of honesty, neatness and accuracy. We will discuss here only the line graph and its use in picturing certain relationships or trends.

The Line Graph in Statistics

Figure 1 shows the population of the United States 1800–1920. It pictures the most significant thing about that population, namely, its accelerating increase. Clearly its chief purpose is

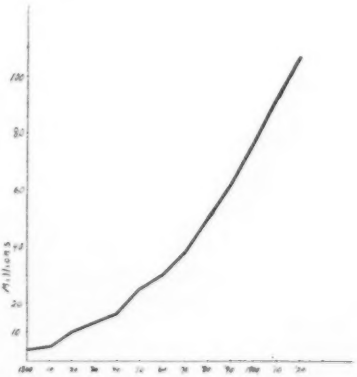


Fig. 1. Population of the United States 1800–1920.

not to provide a means by which to compute the population on June 19, 1893, nor even for the year 1895, but rather to emphasize the relation between the population at any decade and that of the following or preceding decade. The graph might be used to prophesy the population in 1930, but that would be very unreliable and a property merely incidental to the chief purpose of the graph which is to picture the relation between successive ordinates, or to show the trend of the population.

Figure 2 shows the comparative monthly earnings per mile of the steam railroads of the United States. The total revenue, expense and net profit are shown for each month and the graph brings out the seasonal effect on each. Note the increase in September and October. Clearly the graph is a useful and forceful device for representing certain relationships.

Each of these graphs pictures the very simplest relationship, that of additions being made to successive ordinates. Sometimes it is desirable to express that increase in terms of ratio or percent of increase. Fig. 3 represents the same data as Fig. 1 on population. The graph of the same data is entirely different; it was made by plotting the logarithms of the ordinates and shows

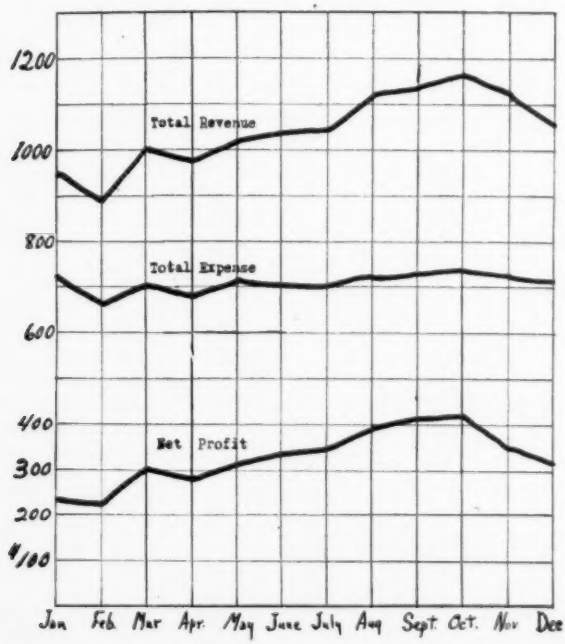


FIG. 2. Comparative monthly earnings per mile of road of steam railroads in the United States. (Brinton, p. 115.)

the ratio between successive years, not the numerical difference.

The steepest part of the graph in Fig. 3 is from 1810–1820, when the population increased only five millions, but from five to ten millions. The steepest part of the graph of Fig. 1 is 1900–1910, when the population increased 16 millions, from 76 to 92 millions. It is clear however that while the numerical increase was less the population actually doubled itself during the earlier decade, and increased by about 20 percent in the later decade.

This type of graph using the logarithms of the ordinates is extremely useful to represent a certain kind of relationship, that

of rate of increase in its strictest sense of ratio or percent of increase. The trend is distinctly downward in Fig. 4; that is, even though the numerical increases were constantly increasing in size, the percent of increase was constantly decreasing in size.

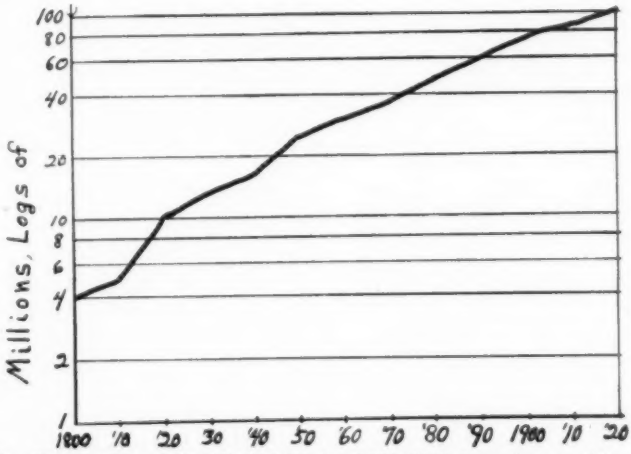


FIG. 3. Population of the United States 1800–1920 by using semi-logarithmic coordinates, thus showing ratio of increase.

May we illustrate by a more simple and fictitious example. Assume a city whose population for five successive years has been 1,000, 2,000, 3,000, 4,000, 5,000. Clearly the increases have been uniform in number, 1,000 each year, but not uniform in percent of increase, because you use a larger and larger base each year in computing percents. The percents of increase would be 100 percent, 50 percent, 33⅓ percent, 25 percent. In order for the percent of increase to be the same each year the population would have had to be 1,000, 2,000, 4,000, 8,000, 16,000. A logarithmic graph of these last figures would be a straight line showing a constant rate of increase, and of the first set of figures a curve approaching a horizontal line and showing a decreasing rate of increase.

Figure 4 is a duplication of Fig. 2 by using logarithms of the ordinates. It shows up the greater percent of increase in the net profits of the railroads, a fact cleverly concealed in Fig. 2.

In a graph using logarithms of the ordinates, the difference

between successive ordinates is the logarithm of the ratio between those ordinates. If this difference increases or decreases, that is, if the ratio varies, it makes the graph a curved line. If that ratio is constant, the graph is a straight line.

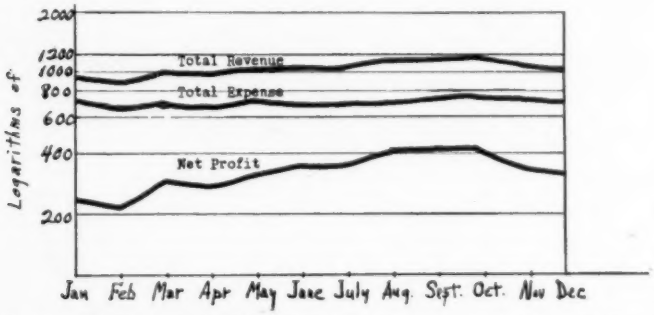


FIG. 4. Comparative monthly earnings per mile of road of steam railroads in the United States, using semi-logarithmic coordinates, thus showing ratio of increase.

What is the purpose in showing these graphs? Simply to support the thesis that the graph is used to picture relationships. These graphs do picture certain very definite relationships, those of differences and ratios between successive elements of a statistical table.

GRAPHS OF FUNCTIONAL RELATIONSHIPS

Figure 5 might also in a sense be considered a statistical graph, but is really the graph of the compound interest formula, $A = P(1 + i)^n$. It shows the relationship between the amount of one dollar placed at interest compounded annually at 5 percent and the time. The logarithmic graph of these data would of course be a straight line since the ratio between successive years is constant. If you multiply the amount at the end of any one year by 1.05, you would get the amount at the end of the next year. This graph illustrates again in a more strictly mathematical sense the value graphs have in picturing these very definite relationships.

The mathematical graph usually considered is the picture of the functional relationship between two variables as ordinarily expressed in a formula or an equation. It is useful not merely for picturing that relationship, but also for actually discovering

that relationship, and frequently for obtaining the maximum or minimum value of one of the variables. It also has use in interpolation where only approximate values are desired as in a graph of numbers and their squares.

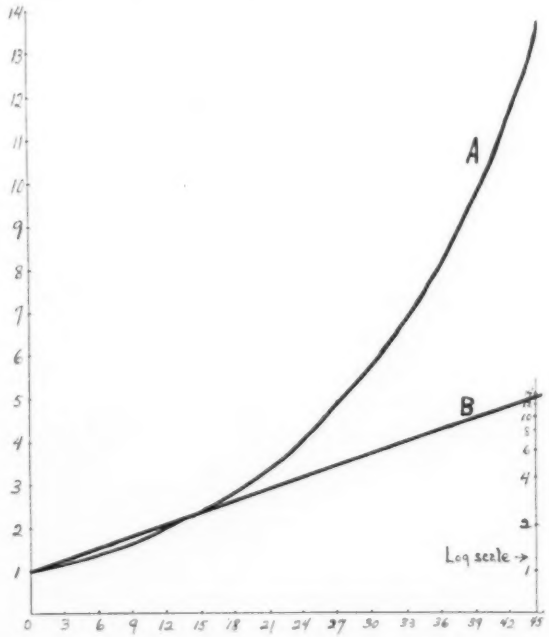


FIG. 5. Amount of $1 at interest compounded annually at 5 percent for 45 years. *B*. Same graph, using logarithms of the ordinates.

Figure 6 is a graph whose points have numbers for their abscissas and the squares of those numbers for the corresponding ordinates. The square root of any number such as 72 is readily obtained with a fair degree of accuracy as about 8.5 or the square of 7.8 is approximately 61.

The making of a graph may help materially in the discovery of a formula expressing the relationship algebraically. Fig. 7 illustrates this by using one of Mr. Betz's problems. The problem is the simple one of finding the relation between the number of sides of a polygon and the number of triangles which can be formed by drawing diagonals from one vertex. A polygon with 4 sides can be divided into 2 triangles; 5 sides, 3 triangles; 6

sides, 4 triangles. Graphing these three points suggests a linear relationship of the nature $T_1 = n - 2$. Checking this formula by plotting other points shows it to be correct.

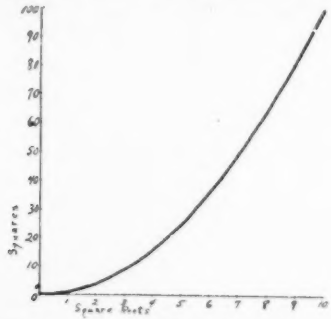


FIG. 6. Numbers and their squares or square roots.

Extend the problem a little and find other relationships. How many diagonals can be drawn from one vertex? From all vertices? How many different triangles are formed by drawing diagonals from all vertices? The first of these relationships after

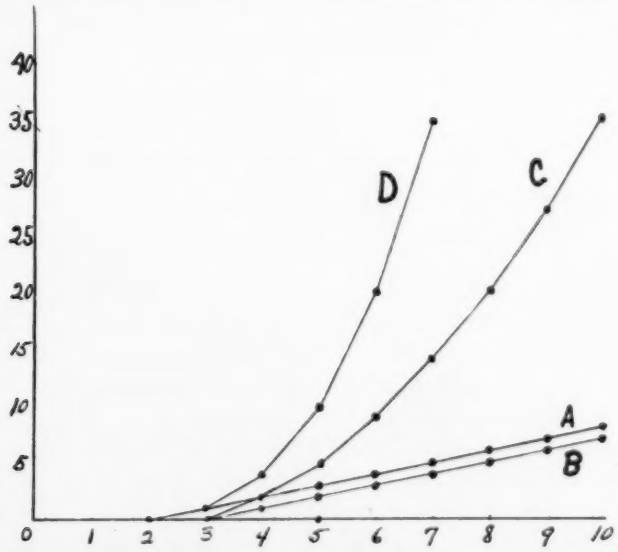


FIG. 7. *A*. Number triangles by diagonals from one vertex. ($T = N - 2$.) *B*. Number diagonals from one vertex of a polygon. ($D = N - 3$.) *C*. Number diagonals from all vertices. ($D = (N - 3)(N/2)$.) *D*. Triangles by diagonals from all vertices. ($T = (N - 1)(N - 2)(N/6)$.)

counting a few special cases seems clearly to be linear, and is then rather easily obtained as $D_1 = n - 3$. The other relation-

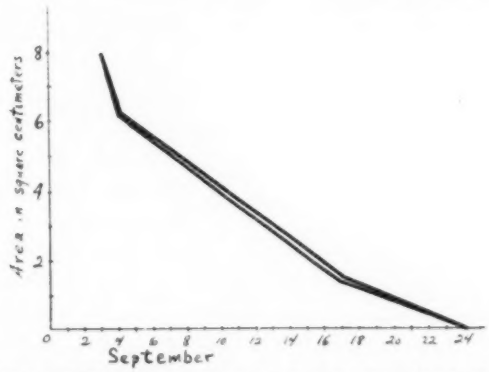


FIG. 8. "Law of Cicatrization" or healing, by Dr. Alexis Carrel, applied to 27-year-old man wounded in War. Upper line is actual healing curve, lower is predicted curve.

ships after plotting a few points are as easily seen to be much less simple and probably to be of a quadratic or cubic form in n. The graph eliminates at once all thought of a linear relationship,

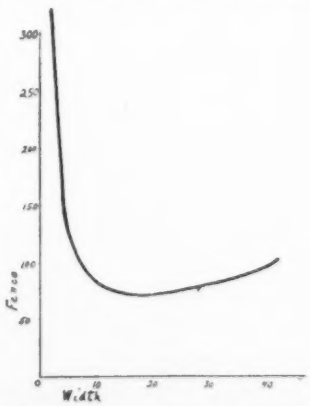


FIG. 9. Fence required to enclose 4 acres of pasture next to a river for various widths of the field.

and careful analysis brings out the other formulas to be $D_n = (n-3)n/2$ and $T_n = (n-1)(n-2)n/6$. The technique of discovering what the formula is after graphing it is very simple

for linear functions and more complicated for curved line relations. The limits of this article do not permit the discussion of the development of formulas from empirical data, its purpose is merely to show that the graph is an extremely valuable helper in that process.

A most significant use was made of the graph during the War. Dr. Alexis Carrel in using his famous method of sterilizing and healing wounds found what he thought to be a regularity in the reduction of bacteria and the healing of the wound which suggested to him the possibility of getting a formula or law which would foretell the date of recovery. He gave his data to a noted mathematician, duNouy, who worked out the formula which expressed the "Law of Cicatrization" or the law of healing. Given the area of the wound and the age of the person he could determine the day the wound would be healed almost as readily as we could find the area of a triangle, given its base and altitude. In Fig. 8 the lower curve is the healing curve as prophesied on the day the patient entered the hospital and the upper one is the actual healing curve.

Maximum and minimum problems can be solved very easily by graphing. While the calculus method is much quicker it also depends on the graph for its analysis. Fig. 9 is the graphic solution of a very real problem. A little study of the situation will show the solution to be remarkably simple and not too difficult for an eighth or ninth grade class if they can make a simple line graph and set up a formula.

Problem 1: Fence four acres next to a river with the least fence. Analysis of problem 1: Data for the solution of this problem by graphing can readily be obtained even without a formula. Since 4 acres equals 640 square rods, if the field is 1 rod wide it would be 640 rods long and require 642 rods of fence for the three sides, no fence being needed along the river. If 2 rods wide, then 320 rods long and 324 rods of fence for three sides, etc., as per Table 1. Clearly 16 or 20 rods wide give the least fence as suggested by the table, but a graph of these data using points whose abscissas are the first column and whose corresponding ordinates are the last column as in Fig. 9 suggests a better value between 16 and 20. Trying 18 we get 71⅑ rods of fence. By calculus methods readers have no doubt already found the best width to be $8\sqrt{5}$ or about 17.89 rods. How-

ever 18 rods or even 16 or 20 rods is probably sufficiently accurate for all practical purposes, and this is readily obtained by a simple analysis of the data and a graph of the results.

TABLE 1

Width	Length	Fence
1	640	642
2	320	324
5	128	138
10	64	84
16	40	72
20	32	72
25	25.6	75.6
32	20	84
40	16	96

The formula for the above solution is $F = 2w + 640/w$ where $F =$ amount of fence needed and w the width of the field. The graph in Fig. 9 is a graph of this formula, but obtained by simple trial and error methods.

Problem 2: Make a box of greatest possible volume by cutting a square out of each corner of a piece of tin 12×20 inches and by then folding up the sides and soldering. Table 2 is an analysis of this problem, and the reader may make or imagine his own graph of the data.

TABLE 2

Depth	Width	Length	Volume
0	12	20	0
1	10	18	180
2	8	16	256
3	6	14	252
4	4	12	192
5	2	10	100
6	0	8	0

Evidently the size square to give the greatest volume lies between 2 and 3 inches. This conclusion may be drawn from the table but in so doing we are more or less unconsciously using the graph. 2.5 as the side of the square gives a volume of 262.5, and 2.4 a volume of 262.6, or by calculus the best value is 2.43. An approximation as close as desired can be obtained by the graph, and of course the calculus method also depends on the graph for its analysis.

Conclusion

You have probably noticed that no mention has been made of negative numbers. They have been left out purposely, but graphing can very readily be used as a project to motivate signed numbers. The intent in this discussion has been to suggest a unit of work in graphing so simple and vital as to have some claim to consideration as a part of the education of all citizens. Except for the theory behind the logarithmic plotting which need not be taught, all of this work is simple, fundamental, unified, provides drill in arithmetic and involves much reasoning, planning and purposeful activity.

The use of graphs to show trends in statistical data, to help the scientific experimenter in deriving formulas, to solve maximum and minimum problems, and to picture all sorts of relationships is a powerful tool in mathematics. As the teaching partner of the function concept textbooks in the near future will use this inseparable and indispensable pair, not as vague, aimless filling, but as the heart of the course in Mathematics.

"The new mathematics is a sort of supplement to language, affording a means of thought about form and quantity and a means of expression, more exact, compact, and ready than ordinary language. The great body of physical science, a great deal of the essential facts of financial science, and endless social and political problems are only accessible and only thinkable to those who have had a sound training in mathematical analysis, and the time may not be far remote when it will be understood that for complete initiation as an efficient citizen of one of the great new complex world wide states that are now developing it is as necessary to be able to compute, to think in averages and maxima and minima, as it is now to be able to read and write."

H. G. Wells, *Mankind in the Making* (London, 1904), pp. 191–192.

THE NEW MATHEMATICS AS A PART OF THE NEW EDUCATION, "ITS NATURE AND FUNCTION"[1]

By WALTER F. DOWNEY

Head Master, The English High School, Boston, Mass.

When we consider the present tendency in secondary mathematics as a reorganized subject in the newer curriculum, we are immediately impressed by the increased stress on the human phase of the subject. During the last quarter of a century, in our constant effort for sound progress, our best guide has been *the pupil*. Mathematics is to-day making a greater contribution because of this changed emphasis. Our present program definitely calls for the adaptation of the subject to the child. Considerations of his interests, tastes, capacities, life situations, and civic needs to-day, as never before, are the determining factors in the revision of curricula, reorganization of courses of study and textbooks, as well as the basic guide in the newer technique of instruction.

Of the influences, in the field of mathematics, that have stimulated and directed the progressive movement, the following have made impressive contributions:

1. The Perry movement in England.
2. The address by Professor E. H. Moore before the American Mathematical Society.
3. The report of the International Commission on the Teaching of Mathematics.
4. The report of the National Committee on Mathematical Requirements.

In the general field of secondary education, the following have exerted some influence on the changes made in mathematics:

1. The Report on the Commission on the Reorganization of Secondary Education.
2. The organization and progress of the junior high school movement.

[1] Address at the Boston Meeting of the National Council of Teachers of Mathematics, February 24, 1928.

3. The nation-wide discussion of curriculum revision.

At a meeting of the Department of Superintendence, of the National Education Association, held in February, 1925, a speaker giving an address on "Curriculum Revision" stated substantially as follows: "The entire curriculum of the secondary schools needs the same type of reorganization as that already made in mathematics under the leadership of the National Committee on Mathematical Requirements." This and other similar evidence will justify the statement that in many ways mathematics has been foremost in the secondary field to accept and apply the viewpoint of the newer education. We may add that the activity in this subject has inspired similar progressive changes in Latin, the sciences, modern languages, as well as other subjects.

Accompanying all of these changes has been a careful and scientific *pupil* study, accounting and analysis. The effect of this work has been the recognition of a second major trend, viz., *the democratization* of mathematics. It is very significant that the needs of *all* children are now of real concern in the field of secondary mathematics.

No longer ago than the year 1911, the report of the International Commission pointed out the following: "The curriculum of mathematics is determined in general by the admission requirements of colleges. This is, of course, confessedly so in the states in which there is a complete and statewide organization of education, with the university recognized as its final stage. It is also true in other states, and even in the smaller communities where few, if any, of the pupils may be planning to go to college, and where the local school committee disclaims the intention of following university guidance; and it is true in the mathematical curriculum where it is not true of the other subjects of study."

In contrast with this statement we note the changed viewpoint as expressed by the National Committee, when we find in its report the following two statements:

1. "The separation of prospective college students from others in the earlier years of the secondary school is neither feasible nor desirable. It is, therefore, obvious that secondary school courses in mathematics cannot be planned with specific reference to college."

2. "The course for these three years (seven, eight, and nine) should be planned as a unit with the purpose of giving each pupil the most valuable mathematical training he is capable of receiving in three years with little reference to courses which he may or may not take in succeeding years."

The College Entrance Examination Board states in a recent document: "It is the purpose of the College Entrance Examination Board to set its requirements in mathematics in such a way that the work of preparation may at the same time be of the greatest intrinsic value to the pupil. . . ."

Without further evidence we may properly observe that the aim in the secondary school to-day is worthwhile mathematics for every child. Formerly the needs of less than 20 percent of the pupils dominated the content of the mathematical curriculum. To-day, the needs of the 80 percent are paramount, while the needs of the other 20 percent are safeguarded. College entrance requirements have been and are being revised on a basis which is more progressive and educationally more democratic than ever before. The articulation of secondary school and college is not everywhere complete, but the common problems are being solved in a very cooperative manner.

Again, as a result of our study of the individual child, the tools of instruction are vastly improved. Have we ever before had textbooks which had as great an appeal to the pupil as those of to-day? The modern textbook differs as widely from the older type as the automobile of to-day from that of twenty-five years ago. The older type has been rendered obsolete by the march of progress. The same is true of every other subject in the curriculum. The secondary school texts to-day are attractive and scholarly. Formerly they were sometimes scholarly, never attractive.

In the class room, the methods now used seek to solve, so far as possible, the complex problems relating to individual differences. In the selection and organization of subject matter, obsolete material has been discarded, meaningful material has been retained. In this selective process, of the subjects in the secondary school, mathematics has led the way.

In the revision of the curriculum the important factor in determining the relative importance of a subject has been the contribution of that subject to the training of efficient, thoughtful,

and useful citizens. There is now a uniformly accepted responsibility among educators to consider every method, every plan, and every subject that will assist in this great aim.

As a group interested in mathematics, let us ask ourselves frankly whether we have reflected often enough on the civic values of our subject. Have we in the secondary field, in our justifiable enthusiasm for our subject, ever almost taken the view that we are teaching the subject through the pupils, instead of the pupils through the subject. Do we, in groups of teachers of other subjects, present the proper claims of mathematics in the upbuilding of present-day citizenship. Are these claims artificial or real and far-reaching?

In the modern conception of education a most important phase of character development lies in guiding the pupil to complete achievement in all of his tasks, so that he may acquire what is sometimes spoken of as achieving power and habit. The process of education through self-activity requires three conditions to operate effectively: (1) that the pupils should be given opportunity to be problem finders as well as problem solvers, because problem finding and solving are infinitely more productive in the development of vital minds than is problem solving alone; (2) that whatever activity is undertaken, whether it be academic study, mechanic arts, practical arts, fine arts, or athletics, the principle should be accepted and followed that if the thing is worth assigning and is properly assigned it is worth MASTERING one hundred percent, not sixty percent or seventy percent only, before passing on to the next bit of work; and (3) that, before considering any problem as completed, the pupil should feel sure in his own mind, through the use of checks and other means, that his work is correct. In this way he reaches that assured success in one undertaking which brings to him great encouragement amounting to a strong motivation for the next undertaking. This is in accord with the fundamental principle that character building is a cumulative process.

The practical applications in the life of the average citizen are generally recognized: (1) the many uses of the formula, the equation and the graph. (2) The general applicability of mathematics to the different fields of engineering in this mechanical age of ours, an age where, in dealing with tariffs, state and federal taxes, the important economic tendencies of the day, the citizen of to-day without an adequate mathematical training is

improperly equipped to assist in solving national and world problems of great importance.

One of the essential needs of every citizen is the ability to form independent judgments based upon real and trustworthy evidence. Can any phase of a subject give more complete and adequate training to this end than the study of a logically constructed geometric proof? Many of you may have had the same experience as the speaker who has repeatedly heard members of the legal profession openly acknowledge their great debt to the study of high school mathematics.

In mathematics we are concerned with the analysis of concrete situations. Life, for the effective citizen, is a succession of concrete situations which must be speedily and properly analyzed and solved. He must learn to do by doing. Not only sound theory but adequate practice is the accepted guide to the successful solution of life situations. We accept the adage that "Necessity is the Mother of Invention." In mathematics we encourage and stimulate the spirit of initiative and of discovery.

In conclusion let me say that we may properly stress the moral training inherent in the proper teaching of mathematics. There is no finer expression of our thought than that found in the conclusion of an address of Doctor Eugene Smith delivered before the Mathematical Association of America, in 1921, entitled "Religio Mathematici."

He concludes:

"And what is the conclusion? Does mathematics make a man religious? Does it give him a basis of ethics? Will the individual love his fellowman more certainly because of the square of the hypotenuse? Such questions are trivial; they are food for the youthful paragrapher. Mathematics makes no such claims. What we may safely assert, however, is this,—that mathematics increases the faith of a man who has faith; that it shows him his finite nature with respect to the Infinite; that it puts him in touch with immortality in the form of mathematical laws that are eternal. . . ."

And so it must be with us,—that we should teach "the science venerable," not merely for its technique, not solely for this little group of laws or that; not only for a body of unrelated propositions or for some examination set by the school, but we should teach it primarily for the beauty of the discipline, for the faith it gives in truth, eternal law, in the Infinite. . . ."

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